1. Evaluate the following integrals. Show your work.

(a) (6 points) $\int_0^1 \frac{x}{x^2+3x+2}dx$

Partial Fractions:

$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$x = A(x+2 \quad B(x+1)$

$-1 = A(1) + \quad (0) \Rightarrow A = -1$

$-2 = A(0) \mid \quad (-1) \Rightarrow B = 2$

$= \int_0^1 \frac{x}{(x+1)(x+2)}\,dx$

$= \int_0^1 \frac{-1}{x+1} + \frac{2}{x+2}\,dx$

$= -\ln|x+1|\Big|_0^1 + 2\ln|x+2|\Big|_0^1$

$= -\ln 2 + \underbrace{\cancel{\ln 1}}_{0} + 2\ln 3 - 2\ln 2$

$= 2\ln 3 - 3\ln 2$

$\left(\begin{array}{l} = \ln 9 - \ln 8 \\ = \ln(9/8) \end{array} \right)$

Answer: $2\ln 3 - 3\ln 2$

(b) (6 points) $\int_0^2 \frac{x}{x^4+2x^2+2}dx$

Substitute $u = x^2$

$\frac{1}{2}du = x\,dx$

$= \int_{u=0}^{u=4} \frac{1}{u^2+2u+2}\,\frac{1}{2}\,du$

$= \frac{1}{2}\int_0^4 \frac{1}{(u+1)^2+1}\,du$

Substitute $v = u+1$

$dv = du$

$= \frac{1}{2}\int_1^5 \frac{1}{v^2+1}\,dv$

$= \frac{1}{2}\arctan v\Big|_1^5 = \frac{1}{2}(\arctan 5 - \overbrace{\arctan 1}^{=\pi/4})$

Answer: $\frac{1}{2}(\arctan 5 - \pi/4)$

2. Evaluate the following integrals. Show your work.

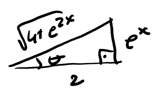
(a) (6 points) $\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$ ① $u = e^x$, $du = e^x\,dx$

$= \int \frac{1}{\sqrt{4+u^2}}\,du$ ② $u = 2\tan\theta$, $du = 2\sec^2\theta\,d\theta$

$= \int \frac{1}{2\sec\theta}\,2\sec^2\theta\,d\theta$

$= \int \sec\theta\,d\theta$

$\tan\theta = \frac{u}{2} = \frac{e^x}{2} = \frac{\text{opp}}{\text{adj}}$



$\Rightarrow \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4+e^{2x}}}{2}$

$= \ln|\tan\theta + \sec\theta| + C$

$= \boxed{\ln\left|\frac{e^x}{2} + \frac{\sqrt{4+e^{2x}}}{2}\right| + C}$

$= \ln\left(e^x + \sqrt{4+e^{2x}}\right) \underbrace{-\ln 2 + C}$

Answer: $\boxed{\ln\left(e^x + \sqrt{4+e^{2x}}\right) + C}$

(b) (6 points) $\int \sec^3(x)\,dx$ (*hint: try integration by parts*)

① By Parts:

$u = \sec x$ $dv = \sec^2 x\,dx$

$du = \tan x \sec x\,dx$ $v = \int \sec^2 x\,dx = \tan x$

$I = \int \underbrace{\sec x}_{u}\,\underbrace{\sec^2 x\,dx}_{dv}$

$= \tan x \sec x - \int \tan^2 x \sec x\,dx$

$= \tan x \sec x - \int (\sec^2 x - 1)\sec x\,dx$

$= \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx$

$I = \tan x \sec x - I + \ln|\tan x + \sec x|$

$I = \frac{1}{2}\left(\tan x \sec x + \ln|\tan x + \sec x|\right) + C$

Answer: $\boxed{\frac{1}{2}\left(\tan x \sec x + \ln|\tan x + \sec x|\right) + C}$

3. (8 points) For each integral below, state which method applies best.

Your answer should be in one of the following forms:

- u-substitution, with $u = ...$ (specify the substitution)
- integration by parts, with $u = ...$, and $dv = ...$ (specify the parts)
- trigonometric substitution, with $x = ...$ (specify the trig sub)
- partial fractions, with fractions: $\frac{A}{(...)} + ...$ (specify the fractions, do not calculate A, etc)

No need to justify or compute anything – and do not evaluate the integrals!

(a) $\int \sin(x)\cos^2(x)\,dx$ Method: u-substitution, $u = \cos x$

(b) $\int x\sec^2(x)\,dx$ Method: Integration by Parts

$u = x$, $dv = \sec^2 x\,dx$

(c) $\int \frac{2x+1}{x^4+x^3+x^2}\,dx$ Method: Partial Fractions

$\int \frac{2x+1}{x^2(x^2+x+1)}$

$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$

(d) $\int \frac{x^2}{(x^2-4)^{3/2}}\,dx$ Method: Trigonometric Substitution

$x = 2\sec\theta$

4. A drone starts from a height of 2 m above the ground at $t = 0$, and flies straight up, with velocity at t seconds, $0 \leq t \leq 10$, given by

$$v_1(t) = 0.3t^2 \, m/s.$$

At $t = 10$ seconds, its battery fails, so from that time on the acceleration acting on the drone is

$$a_2(t) = -9.8 \, m/s^2,$$

causing it to eventually fall down and crash on the ground.

(a) (5 points) What is the drone's height above ground at the moment when its battery fails?

Let $s_1(t)$ denote the position of the drone above the ground during the first 10 seconds.

Then $s_1(0) = 2$ m and

$$s_1(10) - s_1(0) = \int_0^{10} v_1(t)\,dt = \int_0^{10} 0.3t^2\,dt = 0.1t^3\Big|_0^{10} = 0.1(1000) = 100$$

so $s_1(10) = 100 + 2 = 102$ meters

Answer: 102 meters

(b) (5 points) How long does the drone take to crash on the ground, from the time its battery fails? Round your answer to the nearest tenth of a second.

Reset $t = 0$ when the battery fails.

Then: $a_2(t) = -9.8$

$v_2(t) = -9.8t + C$, with $v_2(0) = v_1(10) = 0.3(10^2) = 30$

so $v_2(t) = -9.8t + 30$

The drone's position above ground t seconds after its battery fails is:

$h_2(t) = -4.9t^2 + 30t + D$ with $h_2(0) = 102$ (part (a))

so $h_2(t) = -4.9t^2 + 30t + 102$

It falls on the ground when $h_2(t) = 0$:

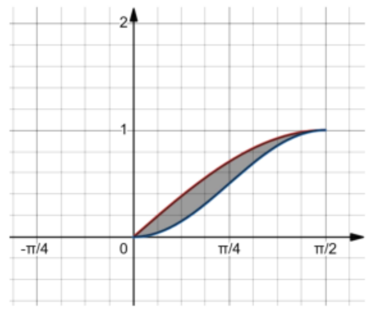
$-4.9t^2 + 30t + 102 = 0$

Quadratic Formula: $t = \dfrac{-30 \pm \sqrt{900 + 4(4.9)(102)}}{2(-4.9)} \approx \begin{cases} -2.43 \\ 8.56 \end{cases}$

Answer: 8.6 seconds

5. (8 points) Find the area bounded by the curves:

$$y = \sin x \text{ and } y = \sin^2 x, \text{ for } 0 \le x \le \pi/2.$$



$$\text{Area} = \int_0^{\pi/2} \sin x - \sin^2 x \, dx$$

$$= \int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} \frac{1-\cos 2x}{2} \, dx$$

$$= \left(-\cos x - \frac{x}{2} + \frac{\sin 2x}{4} \right) \Big|_0^{\pi/2}$$

$$= -\overset{0}{\cancel{\cos \tfrac{\pi}{2}}} + \cos 0 - \frac{\pi/2}{2} + \cancel{0} + \frac{\overset{0}{\cancel{\sin \pi}}}{\cancel{4}} - \frac{\overset{0}{\cancel{\sin 0}}}{\cancel{4}}$$

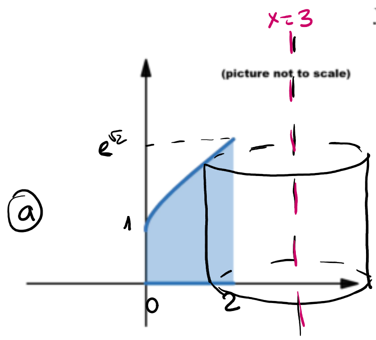
$$= 1 - \pi/4$$

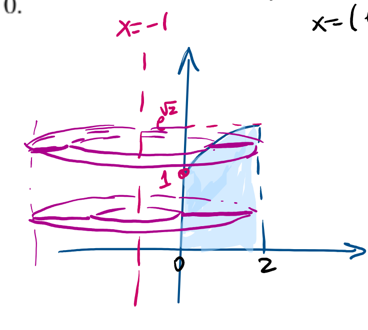
Answer: Area = $1 - \pi/4 = \frac{4-\pi}{4}$

6. Let R be the region in the xy-plane bounded by the curves:

$$y = e^{\sqrt{x}},\ x = 0,\ x = 2 \text{ and } y = 0.$$

$y = e^{\sqrt{x}}$
$\ln y = \sqrt{x}$
$x = (\ln y)^2$





(a) (5 points) **Set up, but do not evaluate,** an integral (or a sum of integrals) equal to the volume of the solid of revolution obtained by revolving R about the vertical line $x = 3$ using the **method of cylindrical shells**.

$$\text{Volume} = \int_0^2 2\pi\,(\text{radius})(\text{height})\,dx$$

$$= \int_0^2 2\pi\,(3-x)\,e^{\sqrt{x}}\,dx$$

Answer: $\int_0^2 2\pi\,(3-x)\,e^{\sqrt{x}}\,dx$

(b) (5 points) **Set up, but do not evaluate,** an integral (or a sum of integrals) equal to the volume of the solid of revolution obtained by revolving R about the vertical line $x = -1$ using the **method of disks and washers**.

The disks/washers would have to be horizontal so the integral has to be in terms of y and separated into $0 \le y \le 1$ and $1 \le y \le e^{\sqrt{2}}$.

or $$9\pi - \pi + \int_1^{e^{\sqrt{2}}} \left[\pi(3)^2 - \pi\,(\ln y)^4\right] dy$$

or $$\underbrace{\int_0^{e^{\sqrt{2}}} \pi(3)^2\,dy}_{(9\pi)e^{\sqrt{2}}} - \underbrace{\int_0^1 \pi(1)^2\,dy}_{\pi} - \int_1^{e^{\sqrt{2}}} \pi\,(\ln y)^4\,dy$$

Answer: $\boxed{\int_0^1 \pi(3)^2 - \pi(1)^2\,dy + \int_1^{e^{\sqrt{2}}} \pi(3)^2 - \pi\left[\ln^2 y\right]^2 dy}$

7. (10 points) Find the mass of an empty leaky bucket given the following information:

- The bucket held 10 liters of water at the ground level and was lifted to the height of 16 meters.
- While being lifted, water leaked out of the bucket at the rate of 1/8 liters per meter.
- The total work done was 1960 J.

Assume that the mass of the rope used to lift the bucket was negligible and can be ignored.
The mass of 1 liter of water is 1 kilogram, and the acceleration due to gravity is 9.8 meters/sec^2.

Let m be the mass of the bucket, in kgs. (constant)

The bucket holds 10 liters of water initially, and loses 1/8 L/m

so, at y meters above ground, it has

$10 - 1/8\, y$ liters of water in it,

which means the mass of water in the bucket is

$10 - 1/8\, y$ kgs. (at y meters)

Total mass: $m + (10 - 1/8\, y)$ kgs

The work done is:

$$1960\text{ J} = \int_0^{16} (\text{Force})\,dy = \int_0^{16} \left(m + 10 - \tfrac{1}{8}y\right)(9.8)\,dy$$

Divide by 9.8 and integrate:

$$\frac{1960}{9.8} = 200 = \left(my + 10y - \frac{1}{16}y^2\right)\Big|_0^{16}$$

$$= \left(16m + 160 - \frac{1}{\cancel{16}}16^{\cancel{2}}\right) - (0)$$

$$= 16m + 144$$

$200 = 16m + 144$

$16m = 56$

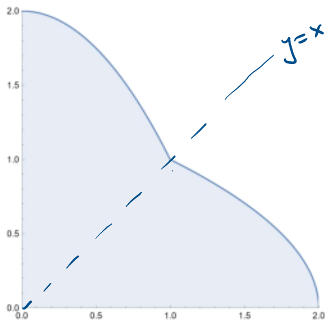
$m = 3.5$

ANSWER: mass of bucket $m =$ 3.5 kg

8. (10 points) Consider the region in the first quadrant bounded below by the x-axis, bounded on the left by the y-axis, bounded above by the graph of $y = 2 - x^2$ for $0 \le x \le 1$, and bounded above by the graph of $y = \sqrt{2-x}$ for $1 \le x \le 2$. See the picture.

Find the center of mass. Give the answer in the exact form.

Note: You can use symmetry but to get full credit, you have to provide a computation supporting the use of symmetry. Saying that the picture looks symmetric is not enough.



The two functions are inverses of each other

$y = 2 - x^2 \iff x^2 = 2 - y \underset{(x \geq 0)}{\iff} x = \sqrt{2-y}$

so their graphs are symmetric relative to the line $y = x$. Hence $\boxed{\bar{x} = \bar{y}}$

1) area $= \int_0^1 (2-x^2)\,dx + \int_1^2 \sqrt{2-x}\,dx \leftarrow$ $u = \sqrt{2-x}$, $u^2 = 2-x$, $2u\,du = -dx$.

$= \left(2x - \frac{1}{3}x^3\right)\Big|_0^1 + \int_1^0 u\,(-2u)\,du$

$= \left(2 - \frac{1}{3}\right) + \int_0^1 2u^2\,du = \frac{5}{3} + \frac{2}{3}u^3\Big|_0^1 = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$

$\boxed{\text{area} = 7/3}$

2) $\bar{y} = \frac{1}{\text{area}}\left[\int_0^1 \frac{1}{2}(2-x^2)^2\,dx + \int_1^2 \frac{1}{2}\left(\sqrt{2-x}\right)^2 dx\right]$

$= \frac{3}{7}\,\frac{1}{2}\left[\int_0^1 4 - 4x^2 + x^4\,dx + \int_1^2 (2-x)\,dx\right]$

$= \frac{3}{7}\,\frac{1}{2}\left[\left(4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2\right]$

$= \frac{3}{7}\,\frac{1}{2}\left[\left(4 - \frac{4}{3} + \frac{1}{5}\right) + \left(\cancel{4} - \cancel{\frac{4}{2}} - \cancel{2} + \frac{1}{2}\right)\right]$

$= \frac{\cancel{3}}{7}\,\frac{1}{2}\left(\frac{120 - 40 + 6 + 15}{(\cancel{3})(5)(2)}\right) = \frac{101}{140}$

ANSWER: $(\bar{x}, \bar{y}) = \left(\frac{101}{140}, \frac{101}{140}\right)$

Alternative Approach:

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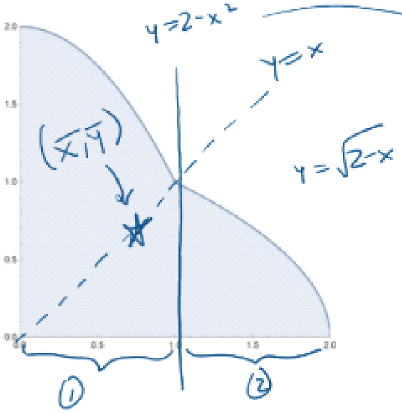
8. (10 points) Consider the region in the first quadrant bounded below by the x-axis, bounded on the left by the y-axis, bounded above by the graph of $y = 2 - x^2$ for $0 \le x \le 1$, and bounded above by the

Alternative Approach

8. (10 points) Consider the region in the first quadrant bounded below by the x-axis, bounded on the left by the y-axis, bounded above by the graph of $y = 2 - x^2$ for $0 \le x \le 1$, and bounded above by the graph of $y = \sqrt{2-x}$ for $1 \le x \le 2$. See the picture.

   Find the center of mass. Give the answer in the exact form.

   Note: You can use symmetry but to get full credit, you have to provide a computation supporting the use of symmetry. Saying that the picture looks symmetric is not enough.



Interchanging x and y, then solving:

$x = 2 - y^2$

$y^2 = 2 - x$

$y = \sqrt{2-x}$

$\therefore$ $y = 2 - x^2$ and $y = \sqrt{2-x}$ are reflections about $y = x$.

$\therefore$ $\bar{x} = \bar{y}$.

① $A_1 = \int_0^1 (2-x^2)\,dx = \left[2x - \frac{x^3}{3}\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$

$\bar{x}_1 = \frac{1}{A_1}\int_0^1 (2-x^2)\,x\,dx = \frac{3}{5}\int_0^1 (2x - x^3)\,dx = \frac{3}{5}\left[x^2 - \frac{x^4}{4}\right]_0^1$

$= \frac{3}{5}\left[1 - \frac{1}{4}\right] = \frac{9}{20}$

② $A_2 = \int_1^2 \sqrt{2-x}\,dx = \left[-\frac{2}{3}(2-x)^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}$

$\bar{x}_2 = \frac{3}{2}\int_1^2 \sqrt{2-x}\,x\,dx = -\frac{3}{2}\int_1^0 \sqrt{u}\,(2-u)\,du = \frac{3}{2}\int_0^1 \left(2\sqrt{u} - u^{\frac{3}{2}}\right)du$

$u = 2 - x$

$du = -dx$

$-du = dx$

$= \frac{3}{2}\left[\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$

$= \frac{3}{2}\left[\frac{4}{3} - \frac{2}{5}\right]$

$\therefore \bar{x} = \frac{m_1\bar{x}_1 + m_2\bar{x}_2}{m_1 + m_2} = \frac{\frac{3}{4} + \frac{4}{3} - \frac{2}{5}}{A_1 + A_2}$

ANSWER: $(\bar{x}, \bar{y}) =$ $\boxed{\left(\frac{101}{140}, \frac{101}{140}\right)}$

9. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2y^2, \quad y(0) = 1.$$

For full credit, write your answer in explicit form, $y = f(t)$, and simplify it.

$$\frac{dy}{dt} = 1+t^2+y^2(1+t^2) = (1+t^2)(1+y^2)$$

$$\int \frac{1}{1+y^2}\,dy = \int (1+t^2)\,dt$$

$$\arctan y = t + \frac{1}{3}t^3 + C$$

$$y(0)=1: \quad \arctan 1 = \frac{\pi}{4} = 0 + \frac{1}{3}(0)^3 + C \Rightarrow C = \pi/4$$

$$\arctan y = t + \frac{1}{3}t^3 + \pi/4$$

$$y = \tan\left(t + \frac{1}{3}t^3 + \pi/4\right)$$

ANSWER: $y =$ $\tan\left(t + 1/3\, t^3 + \pi/4\right)$

10. A pan of lasagna has an internal temperature of 40° F at the time when it is placed in an oven whose temperature is kept constant at 380° F.

(a) (3 points) Newton's Law of Cooling states that the rate of cooling (or heating) of an object is proportional to the temperature difference between the object and its surroundings. Use this to write a differential equation and an initial condition for the internal temperature $y(t)$ of the lasagna at t hours after it was placed in the oven. Your equation should involve an unknown proportionality constant k.

Differential Equation: $\frac{dy}{dt} = k(y-380)$ and $y(0) =$ 40

(b) (5 points) After a half hour in the oven, the internal temperature of the lasagna reaches 140° F. Compute the constant k and find the temperature y as a function of time t, in hours.

$$\int \frac{1}{y-380}\,dy = \int k\,dt \Rightarrow \ln|y-380| = kt + C$$

$$y - 380 = C_1 e^{kt} \quad (C_1 = \pm e^C)$$

$$y = 380 + C_1 e^{kt}$$

① (initially: $t=0$, $y(0)=40$ $\Rightarrow 40 = 380 + C_1 \Rightarrow C_1 = -340$) $\Rightarrow y = 380 - 340e^{kt}$

② at $t = 1/2$ $y = 140$ $\Rightarrow 140 = 380 - 340e^{1/2 k}$

$\Rightarrow +240 = +340\, e^{1/2 k}$

$\Rightarrow e^{1/2 k} = \frac{240}{340} \Rightarrow \frac{1}{2}k = \ln\left(\frac{240}{340}\right) \Rightarrow k = 2\ln\left(\frac{240}{340}\right)$

$= 2\ln(12/17)$

Answer: $y(t) =$ $\boxed{380 - 340e^{2\ln(12/17)t}}$

(c) (2 points) The lasagna is finished baking when its internal temperature reaches 165° F. When will this happen?

$$380 - 340\, e^{2\ln(12/17)t} = 165$$

$$e^{2\ln(12/17)t} = \frac{215}{340}$$

$$t = \frac{\ln(215/340)}{2\ln(12/17)} \cong 0.6579\ldots$$

Answer: At $t =$ $\boxed{0.66}$ hours

(round to the nearest two decimals)