1. Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) (5 points) $\int \left(\frac{1}{x-x^2} - \frac{2x}{x-x^2}\right) dx$ $= \int \frac{1-2x}{x-x^2}\,dx$ $\quad u = x - x^2$, $du = (1-2x)\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C = \boxed{\ln|x-x^2| + C}$

OR Partial Fractions $\frac{1-2x}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{(B-A)x + A}{x(1-x)}$

$\begin{cases} -2 = B - A \\ 1 = A \end{cases} \Rightarrow B = -2 + A = -2 + 1 = -1$

$\int \frac{1-2x}{x-x^2}\,dx = \int \frac{1}{x} - \frac{1}{1-x}\,dx = \int \frac{1}{x} + \frac{1}{x-1}\,dx = \boxed{\ln|x| + \ln|x-1| + C}$

(b) (5 points) $\int \cos(2x)e^x\,dx$ $= I = ?$

① IBP: $u = \cos(2x)$, $dv = e^x dx$
$du = -2\sin(2x)\,dx$, $v = e^x$

$I = e^x\cos(2x) + \int 2e^x \sin(2x)\,dx$

② IBP: $u = \sin(2x)$ $dv = 2e^x dx$
$du = 2\cos(2x)dx$ $v = 2e^x$

$= e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx$

Equation in $I = \int \cos(2x)e^x\,dx$:

$I = e^x\cos(2x) + 2e^x\sin(2x) - 4I$

$5I = e^x\cos(2x) + 2e^x\sin(2x) + C$

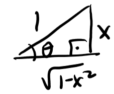
$$\boxed{I = \frac{1}{5}\left(e^x\cos(2x) + 2e^x\sin(2x)\right) + C}$$

2. Evaluate the following integrals. Show all work. Simplify and box your answers.

(a) (5 points) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

Trig Sub: $x = \sin\theta$
$dx = \cos\theta\, d\theta$

Bounds:
$x = 0 \Rightarrow \theta = 0$
$x = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$



$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_0^{\pi/6} \sin^2\theta\, d\theta = \int_0^{\pi/6} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2}\theta\Big|_0^{\pi/6} - \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/6} = \frac{\pi}{12} - \frac{1}{4}\sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}$$

Without bounds: $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \ldots = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\theta + \sin\theta\cos\theta + C$

$$= \frac{1}{2}\arcsin(x) + x\sqrt{1-x^2} + C$$

(b) (5 points) $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

① Sub: $u = \sqrt{x+4} \Rightarrow u^2 = x+4 \Rightarrow x = u^2 - 4$
$2u\,du = dx$.

$$= \int \frac{2\cancel{u}\,du}{(u^2-4)\cancel{u}}$$

$$= \int \frac{2}{(u-2)(u+2)}\,du$$

② Partial Fractions: $\displaystyle\frac{2}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$

$$= \frac{(A+B)u + 2A - 2B}{(u-2)(u+2)}$$

$$2 = \underbrace{(A+B)}_{=0}u + 2\underbrace{(A-B)}_{=1}$$

$A = -B$ OR $B = -A$
& $A - B = A - (-A) = 2A = 1$

$$= \int \frac{1/2}{u-2} + \frac{-1/2}{u+2}\,du$$

$$= \frac{1}{2}\ln|u-2| - \frac{1}{2}\ln|u+2| + C$$

$$= \frac{1}{2}\ln\left|\frac{u-2}{u+2}\right| + C$$

$$= \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C}$$

3. A particle moves along a straight line with velocity $v(t) = 3\cos(2t)$.

(a) (5 points) Compute the particle's displacement from $t = 0$ to $t = 3\pi/4$.

displacement $\Delta x = \int_{t_0}^{t_1} v(t)\,dt = \int_0^{3\pi/4} 3\cos(2t)\,dt$

$= \frac{3}{2} \sin(2t) \Big|_0^{3\pi/4}$

$= \frac{3}{2} \Big[\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} - \underbrace{\sin(0)}_{=0} \Big]$

$= \boxed{-\frac{3}{2}}$

(b) (5 points) Compute the total distance that the particle travels from $t = 0$ to $t = 3\pi/4$.

$= \int_{t_0}^{t_1} |v(t)|\,dt = \int_0^{3\pi/4} 3|\cos(2t)|\,dt$

$\left(\begin{array}{l} \cos 2t = 0 \\ 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \\ t = \frac{\pi}{4}, \frac{3\pi}{4}, \dots \end{array} \right)$

$= \int_0^{\pi/4} 3\cos(2t)\,dt + \int_{\pi/4}^{3\pi/4} 3(-\cos(2t))\,dt$

$= \frac{3}{2} \sin(2t) \Big|_0^{\pi/4} - \frac{3}{2} \sin(2t) \Big|_{\pi/4}^{3\pi/4}$

$= \frac{3}{2}(1-0) - \frac{3}{2}(-1-1)$

$= \frac{3}{2} + 3$

$= \boxed{\frac{9}{2} = 4.5}$

4. (10 points) Determine the arc length of the curve:

$$y = \frac{x^3}{3} + \frac{1}{4x} - 5$$

from $x = \frac{1}{4}$ to $x = 1$. Give the answer in exact simplified form.

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$L = \int_{1/4}^{1} \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx = \int_{1/4}^{1} \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx$$

$$= \int_{1/4}^{1} \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_{1/4}^{1} \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_{1/4}^{1} x^2 + \frac{1}{4x^2}\, dx = \left(\frac{1}{3}x^3 + \frac{1}{4}\left(\frac{-1}{x}\right)\right)\Big|_{1/4}^{1}$$

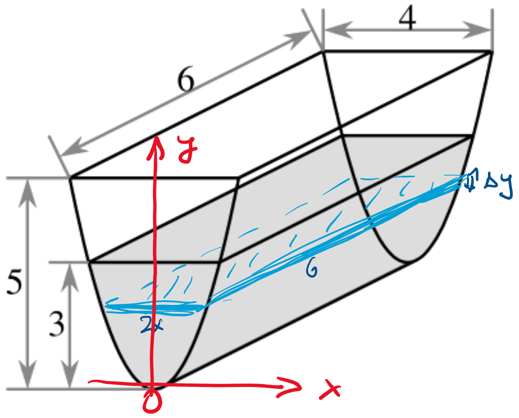
$$= \frac{1}{3}\left(1 - \frac{1}{64}\right) + \frac{1}{4}\left(\frac{1}{1/4} - \frac{1}{1}\right) = \frac{\cancel{63}^{21}}{\cancel{3}(64)} + \frac{1}{4}(4-1)$$

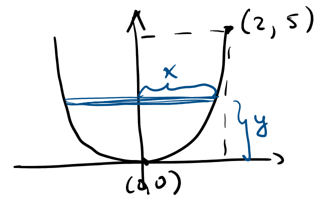
$$= \frac{21}{64} + \frac{3}{4} = \frac{21+48}{64} = \boxed{\frac{69}{64}}.$$

5. (10 points) The tank shown below (all dimensions are in meters) has parabolic ends and vertical cross-sections. The tank is filled with water to a depth of 3 meters. Suppose $\rho = 1000$ kg/m^3 is the density of water and $g = 9.8$ m/s^2 is the acceleration due to gravity.

Give an integral expressing the work done in pumping all of the water to the top of the tank.

Do NOT evaluate the integral. Just set it up.





$y = ax^2$

$(2,5) \Rightarrow 5 = a(4) \Rightarrow a = 5/4$

$\boxed{y = \frac{5}{4}x^2} = 1.25x^2$

A thin layer of water at y meters from the bottom of thickness Δy:

has dimensions $(2x) \times (6) \times \Delta y$ with $x^2 = \frac{4}{5}y$ so $x = \sqrt{\frac{4y}{5}}$

so $\Delta V = 2\left(\sqrt{\frac{4y}{5}}\right)(6)\,\Delta y = 12\sqrt{\frac{4y}{5}}\,\Delta y$ (in cubic meters)

so it requires a force $\rho g\,\Delta V = 9800\left(12\sqrt{\frac{4y}{5}}\right)\Delta y$ (Newtons)

to lift to the top of the tank, i.e. $5-y$ meters.

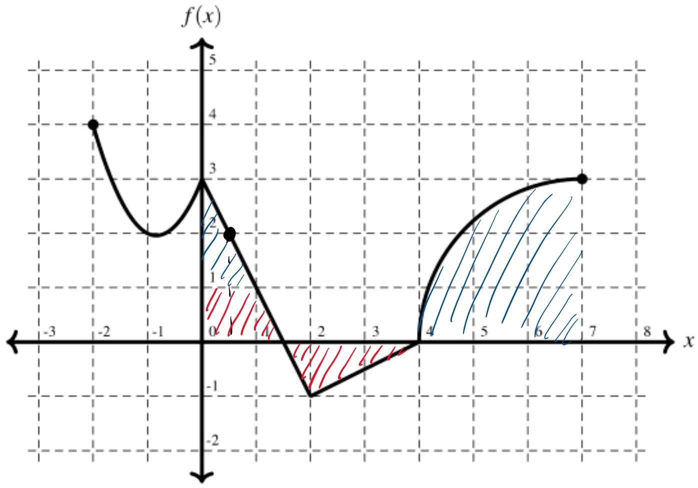
Hence the work for just that layer is

$\approxeq (9800)\left(12\sqrt{\frac{4y}{5}}\right)(5-y)\,\Delta y$ (Joules)

Adding up all the water layers, for $0 \le y \le 3$ meters we get the Riemann Sum whose limit equals:

$$\boxed{W = \int_0^3 (9800)\left(12\sqrt{\frac{4y}{5}}\right)(5-y)\,dy.}$$ Joules

6. (10 points) Here's the graph of a function $f(x)$, consisting of a portion of a parabola, two line segments, and a quarter circle. Use it to answer the questions below. You may use well-known formulas for areas of geometric objects.



(a) (4 points) Compute $\lim_{n\to\infty}\sum_{i=0}^{n}\left(f\left(\frac{7i}{n}\right)\frac{7}{n}\right)$. $= \int_0^7 f(x)\,dx = \frac{1}{2}(3)(1.5) - \frac{1}{2}(1)(2.5) + \frac{1}{4}\pi(3)^2$

$= \boxed{1 + \frac{9\pi}{4}}$

x_i $(0+i\Delta x)$

$\Delta x = \frac{7-0}{n} = \frac{b-a}{n}$

$\Rightarrow [a,b] = [0,7]$.

(b) (4 points) Let $g(x) = \int_{\sin(\pi x)}^{5} f(t)\,dt$. Find $g'\left(\frac{1}{6}\right)$.

$g'(x) = \frac{d}{dx}\left(-\int_5^{\sin \pi x} f(t)dt\right) = -f(\sin(\pi t))\cos(\pi t)\cdot\pi$ (FTC + Chain)

$g'(1/6) = -f(\sin(\pi/6))\cos\left(\frac{\pi}{6}\right)\cdot\pi = -\underbrace{f(1/2)}_{=\cancel{2}}\frac{\sqrt{3}}{\cancel{2}}\cdot\pi = \boxed{-\pi\sqrt{3}}$

(c) (2 points) The Trapezoidal Rule approximation T_3 of $\int_{-2}^{0} f(x)\,dx$ is (circle one):

(i) an underestimate or (ii) an overestimate

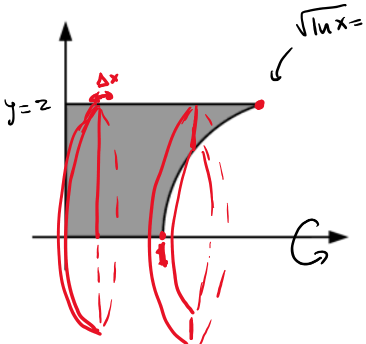
because the function is (circle one):

(a) positive (b) negative (c) increasing (d) decreasing (e) concave up (f) concave down

7. (10 points) Let $\mathcal{R}$ be the region in the first quadrant bounded by $y = \sqrt{\ln(x)}$, $y = 2$, and the two axes. Let $\mathcal{S}$ be the solid formed by revolving $\mathcal{R}$ about the x-axis. (Pictures are not to scale.)

(a) (3 points) Set up an integral expression to evaluate the volume of $\mathcal{S}$ using the **washer method**. Do NOT evaluate the integral expression. Just set it up.

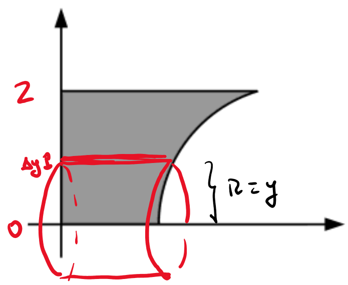
$\sqrt{\ln x} = 2 \Rightarrow \ln x = 4$
$x = e^4$



$$V = \int_0^1 \pi R_1^2\, dx + \int_1^{e^4} \pi R_2^2 - \pi r_2^2\, dx$$

$$= \boxed{\int_0^1 \pi (2)^2 dx + \int_1^{e^4} \pi (2)^2 - \pi \left(\sqrt{\ln x}\right)^2 dx}$$

(b) (3 points) Set up an integral expression to evaluate the volume of $\mathcal{S}$ using the **shell method**. Do NOT evaluate the integral expression. Just set it up.



$y = \sqrt{\ln x}$
$y^2 = \ln x$
$x = e^{y^2}$

$$V = \int_0^2 2\pi R h\, dy$$

$$= \boxed{\int_0^2 2\pi y \left(e^{y^2}\right) dy}$$

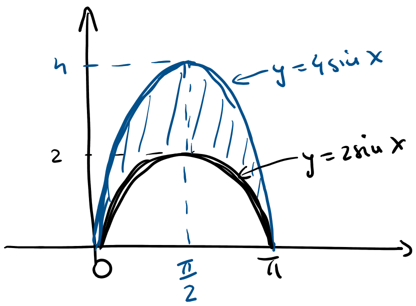
(c) (4 points) Find the volume of $\mathcal{S}$. (Use whichever method from parts (a) or (b) you prefer.)

(a)
$$V = \int_0^1 4\pi\, dx + \int_1^{e^4} 4\pi - \pi \ln x\, dx$$
$$= \cancel{4\pi} + (4\pi e^4 - \cancel{4\pi}) - \pi (x \ln x - x)\Big|_1^{e^4}$$
$$= 4\pi e^4 - \pi\left[(e^4(4) - e^4) - (-1)\right]$$
$$= 4\pi e^4 - \pi(3e^4) - \pi$$
$$= \boxed{\pi e^4 - \pi}$$

(b)
$$V = \int_0^2 2\pi y\, e^{y^2} dy \qquad \left[\begin{array}{l} u = y^2 \\ du = 2y\,dy \end{array}\right]$$
$$= \int_0^4 \pi e^u\, du$$
$$= \pi e^4 - \pi e^0$$
$$= \boxed{\pi e^4 - \pi}$$

(Solutions are given for both methods; either one alone was sufficient)

8. (10 points) Let R be the region bounded by the curves $y = 4\sin x$ and $y = 2\sin x$ and between $x = 0$ and $x = \pi$. By symmetry, we can tell that the x-coordinate of the centroid (center of mass) of R is $\pi/2$. Find the y-coordinate of the centroid of R.



① Area:

$$A = \int_0^\pi 4\sin x - 2\sin x \, dx$$

$$= 2\int_0^\pi \sin x \, dx = -2\cos x \Big|_0^\pi$$

$$= -2(-1-1) = \boxed{4}$$

②

$$M_x = \int_0^\pi \frac{1}{2}(4\sin x)^2 - \frac{1}{2}(2\sin x)^2 \, dx$$

$$= \int_0^\pi 8\sin^2 x - 2\sin^2 x \, dx = \int_0^\pi 6\sin^2 x \, dx = \int_0^\pi 6 \, \frac{1-\cos(2x)}{2} dx$$

$$= 3\int_0^\pi 1 - \cos(2x) \, dx = 3\left(x - \frac{\sin(2x)}{2}\right)\Big|_0^\pi$$

$$= 3\left[(\pi - 0) - (0-0)\right] = 3\pi$$

③ $\bar{y} = \dfrac{M_x}{A} \Rightarrow \boxed{\bar{y} = \dfrac{3\pi}{4}}$

9. (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{y+3}{x\ln x}$$

that satisfies the initial condition:

$$y(e^5) = 27.$$

For full credit, write your answer in explicit form, $y = f(x)$.

Separate variables: $\int \frac{1}{y+3}\,dy = \int \frac{1}{x\ln x}\,dx \leftarrow \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right]$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + c$$

$$\ln|y+3| = \ln|\ln x| + C$$

$$e^{\ln|y+3|} = e^{\ln|\ln x|} \cdot \underbrace{e^{c}}_{c_1}$$

$$|y+3| = |\ln x| \cdot C_1$$

$C_2 = \pm C_1$

$$y+3 = C_2(\ln x)$$

Initial condition:
$x = e^5$, $y = 27$:
$30 = C_2(5) \Rightarrow C_2 = 6$

$$y+3 = 6\ln x$$

$$\boxed{y = 6\ln x - 3}$$

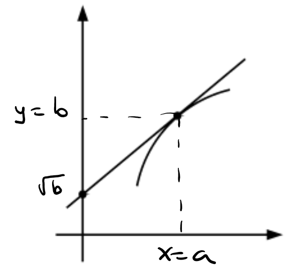
10. (10 points) A certain curve $y = f(x)$ has the following property: (at all $x > 0$)

The tangent line to the curve $y = f(x)$ at any point (a,b) on the curve intersects the y-axis at $(0,\sqrt{b})$. ← $b > 0$.

(a) (5 points) Find a differential equation satisfied by $f(x)$.

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \frac{b-\sqrt{b}}{a-0}$$

$$\boxed{\frac{dy}{dx} = \frac{y-\sqrt{y}}{x}}$$



(b) (5 points) Suppose the curve passes through the point $(1,16)$. Find the function $f(x)$ by solving the differential equation you found in (a).

$$\int \frac{1}{y-\sqrt{y}}\,dy = \int \frac{1}{x}\,dx$$

$u^2 = y$, $2u\,du = dy$

$$\int \frac{1}{u^2-u}\,2u\,du = \int \frac{1}{x}\,dx$$

$$\int \frac{2}{u-1}\,du = \int \frac{1}{x}\,dx$$

$$2\ln|u-1| = \ln|x| + C$$

$$2\ln|\sqrt{y}-1| = \ln|x| + C$$

$$e^{\ln(\sqrt{y}-1)^2} = e^{\ln|x|} \cdot e^{C}$$

$$(\sqrt{y}-1)^2 = C_1 x$$

$x=1$, $y=16$ $\Rightarrow (4-1)^2 = C_1(1) \Rightarrow C_1 = 9$

$$(\sqrt{y}-1)^2 = 9x$$

$$\sqrt{y}-1 = 3\sqrt{x}$$

$$\rightarrow \sqrt{y} = 3\sqrt{x}+1$$

$$\boxed{y = \left(3\sqrt{x}+1\right)^2}$$