1. (12 points) Evaluate the following integrals. Box your final answer.

(a) (6 points) $\displaystyle\int \frac{\sin^2(x)\tan(x)}{\sec(x)}\,dx$

$$= \int \frac{\sin^2(x)\,\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}}\,dx = \int \sin^3(x)\,dx$$

$$= \int (1-\cos^2(x))\sin(x)\,dx$$

$$u = \cos x \qquad du = -\sin x\,dx$$

$$= \int (u^2-1)\,du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \boxed{\frac{1}{3}\cos^3(x) - \cos(x) + C}$$

(b) (6 points) $\displaystyle\int \frac{x^3+2x^2+4x+6}{x^3+3x}\,dx = \int \frac{x^3+3x + 2x^2+x+6}{x^3+3x}\,dx$

$$= \int 1 + \frac{2x^2+x+6}{x(x^2+3)}\,dx$$

P.F.

$$\frac{2x^2+x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$2x^2+x+6 = (A+B)x^2 + Cx + 3A$$

$$\begin{cases} 3A = 6 \Rightarrow A = 2 \\ C = 1 \\ A+B = 2 \Rightarrow B = 0 \end{cases}$$

$$= \int 1 + \frac{2}{x} + \frac{1}{x^3+3}\,dx$$

$$= \boxed{x + 2\ln|x| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$

2. (12 points)

Evaluate the following definite integrals. Simplify your answer, but leave it in exact form.

(a) (6 points) $\int_1^4 \sqrt{\sqrt{x}-1}\,dx$

$\boxed{u\text{-sub: } u=\sqrt{x}-1}$ so $\sqrt{x}=u+1$

$x=(u+1)^2$

$\boxed{dx=2(u+1)\,du}$

$$= \int_0^1 \sqrt{u}\cdot 2(u+1)\,du$$

$$= 2\int_0^1 u^{3/2}+u^{1/2}\,du = 2\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)\Big|_0^1$$

$$= 2\left(\frac{2}{5}+\frac{2}{3}\right) = \boxed{\frac{32}{15}}$$

(b) (6 points) $\int_{-2}^{2} \frac{x^2}{\sqrt{16-x^2}}\,dx$

Trig Sub: $\boxed{\begin{array}{l} x=4\sin\theta \\ dx=4\cos\theta\,d\theta \end{array}}$

Bounds: $x=-2=4\sin\theta \Rightarrow \sin\theta=-\frac{1}{2}$

$\Rightarrow \theta=-\pi/6$

$x=2=4\sin\theta \Rightarrow \sin\theta=\frac{1}{2}$

$\Rightarrow \theta=\pi/6$

$$\int_{-2}^{2}\frac{x^2}{\sqrt{16-x^2}}\,dx = \int_{-\pi/6}^{\pi/6}\frac{16\sin^2(\theta)}{\cancel{\sqrt{16-16\sin^2\theta}}}\cancel{4\cos\theta}\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} 16\sin^2\theta\,d\theta = \int_{-\pi/6}^{\pi/6} 8\left(1-\cos(2\theta)\right)d\theta$$

$$= \left(8\theta - 4\sin(2\theta)\right)\Big|_{-\pi/6}^{\pi/6} = \boxed{\frac{8\pi}{3}-4\sqrt{3}}$$

3. (10 points) Evaluate the following improper integral showing all the appropriate steps. If the integral diverges, then say so.

$$\int_0^\infty xe^{-2x}\,dx$$

$$= \lim_{t\to\infty} \int_0^t x e^{-2x}\,dx$$

$\underline{\text{IBP}}$: $u = x$ $\quad dv = e^{-2x}dx$

$du = dx$ $\quad v = -\frac{1}{2}e^{-2x}$

$$= \lim_{t\to\infty} \left(-\tfrac{1}{2}x e^{-2x}\Big|_0^t - \int_0^t -\tfrac{1}{2}e^{-2x}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-\tfrac{1}{2}t e^{-2t} - \tfrac{1}{4} e^{-2x}\Big|_0^t \right)$$

$$= \lim_{t\to\infty} \left(-\tfrac{1}{2}t e^{-2t} - \tfrac{1}{4}e^{-2t} + \tfrac{1}{4} \right)$$

$$= -\frac{1}{2}\underbrace{\lim_{t\to\infty} \frac{t}{e^{2t}}}_{\frac{\infty}{\infty},\ \text{L'Hospital}} - \frac{1}{4}\lim_{t\to\infty} e^{-2t} + \frac{1}{4}$$

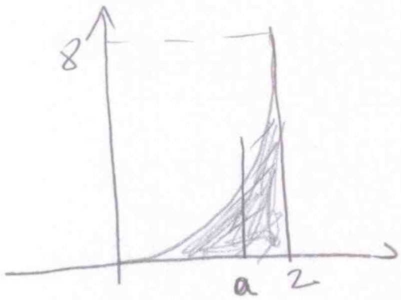
$$= -\frac{1}{2}\lim_{t\to\infty} \frac{1}{2e^{2t}} - \frac{1}{4}(0) + \frac{1}{4}$$

$$= -\frac{1}{2}(0) - \frac{1}{4}(0) + \frac{1}{4}$$

$$= \boxed{\frac{1}{4}}$$

4. (12 points) Consider the region, R, bounded by the curve $y = x^3$, the **vertical** line $x = 2$, and the x-axis.

(a) (6 points) Find the value of the constant a such that the **vertical** line $x = a$ divides the region R into two regions of equal area.



Total Area $= \int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = 4$

Want: $\underbrace{\int_0^a x^3\,dx}_{\frac{1}{4}a^4} = \frac{1}{2}(4) = 2$

$\frac{1}{4}a^4 = 2$

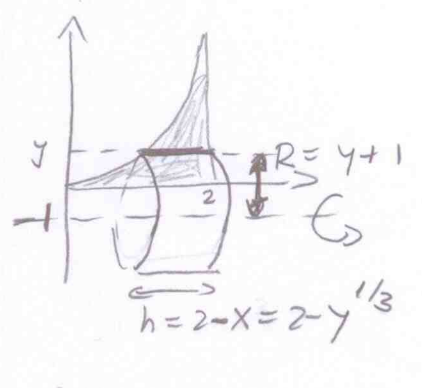
$a^4 = 8$

$\boxed{a = 8^{1/4} = \sqrt[4]{8}}$

(b) (6 points) A solid is obtained by rotating the region R around the **horizontal** line $y = -1$. Set up the integrals you get for the volume of this solid using BOTH the method of cylindrical shells and the method of washers (DO NOT EVALUATE).

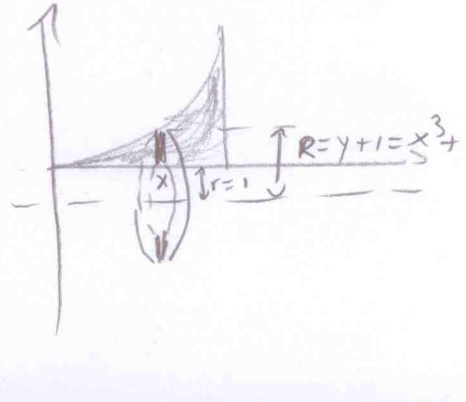
Shells: $\boxed{\int_0^8 2\pi(y+1)(2-y^{1/3})\,dy}$

integral in y



Washers: $\boxed{\int_0^2 \pi(x^3+1)^2 - \pi(1)^2\,dx}$

integral in x



5. (12 points) A cable that weighs 2 lbs per foot is used to lift a bucket of water from a well. The bucket of water weighs 20 lbs, and it needs to be lifted 10 feet to reach the top of the well.

   How far is the bucket from **the bottom of the well** when only half of the total work was done?

Let y denote the height from the bottom of the well.

Then the length of the cable is $10-y$, so the force applied at that height is

$$F(y) = (10-y)\,\cancel{ft}\,(2\,lbs/\cancel{ft}) + 20\text{ lbs}$$

Total work: $W = \int_0^{10} 2(10-y)+20\,dy = 300$ ft-lbs.

Work to lift bucket to b feet from the bottom:

$$\int_0^b 2(10-y)+20\,dy = (40y-y^2)\Big|_0^b = 40b-b^2$$

Want: $40b-b^2 = \frac{1}{2}(300) = 150$

$$b^2-40b+150=0$$

Quadratic Formula: $b = \dfrac{40 \pm \sqrt{1000}}{2} = 20 \pm 5\sqrt{10}$

b must be ≤ 10 ft, so $\boxed{b = 20-5\sqrt{10}\text{ ft}} \cong 4.1886$ ft