1. Evaluate the following integrals. Show your work and box your final answer.

(a) (5 points) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

<u>Integration By Parts</u>: $u = \ln x$, $du = 1/x\,dx$ and $dv = x^{-2}\,dx$, $V = -x^{-1} = \frac{1}{x}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = \boxed{-\frac{\ln x}{x} - \frac{1}{x} + C}$$

(b) (5 points) $\displaystyle\int \frac{8}{(x-2)(x^2+4)}\,dx$

<u>Partial Fractions</u>:

$$\frac{8}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$8 = A(x^2+4) + (Bx+C)(x-2)$$
$$= Ax^2 + 4A + Bx^2 - 2Bx + Cx - 2C$$

$$\Leftrightarrow (A+B)x^2 + (C-2B)x + 4A - 2C = 8$$

$$\Leftrightarrow \begin{cases} A+B=0 & \Rightarrow B=-A \\ C-2B=0 & \Rightarrow C=2B=-2A \\ 4A-2C=8 & \Rightarrow \underbrace{4A-2(-2A)}_{8A}=8 \end{cases}$$

$$\Leftrightarrow \begin{cases} A=1 \\ B=-1 \\ C=-2 \end{cases}$$

$$= \int \frac{1}{x-2}\,dx - \int \frac{x+2}{x^2+4}$$

$$= \ln|x-2| - \underbrace{\int \frac{x}{x^2+4}\,dx}_{\left(\substack{u=x^2+4 \\ \frac{1}{2}du = x\,dx}\right)} - 2\int \frac{1}{x^2+4}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+4| - \cancel{2}\frac{1}{\cancel{2}}\arctan\left(\frac{x}{2}\right) + c$$

$$= \boxed{\ln|x-2| - \frac{1}{2}\ln(x^2+4) - \arctan\left(\frac{x}{2}\right) + C}$$

2. Answer the following questions. Show your work and box your final answer.

(a) (5 points) Evaluate $\int_0^{\sqrt{3}/2} x^2 \sin^{-1}(x)\,dx$

① IBP: $u = \sin^{-1}(x)$, $dv = x^2 dx$
$du = \frac{dx}{\sqrt{1-x^2}}$, $v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \sin^{-1}(x)\Big|_0^{\sqrt{3}/2} - \frac{1}{3}\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{\cancel{3}}\frac{\cancel{3}\sqrt{3}}{8}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{3}\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}\,x\,dx$$

② $u = \sqrt{1-x^2} \Rightarrow u^2 = 1-x^2$
$\Rightarrow x^2 = 1-u^2$
$\cancel{2}x\,dx = -\cancel{2}u\,du$

$x=0$: $u = \sqrt{1-x^2} = 1$
$x = \sqrt{3}/2$ $u = \sqrt{1-3/4} = 1/2$

$$= \frac{\sqrt{3}}{8}\,\frac{\pi}{3} - \frac{1}{3}\int_1^{1/2} \frac{1-u^2}{\cancel{u}}(-\cancel{u})\,du$$

$$= \frac{\pi\sqrt{3}}{24} + \frac{1}{3}\left[u - \frac{1}{3}u^3\right]\Big|_1^{1/2}$$

$$= \frac{\pi\sqrt{3}}{24} + \frac{1}{3}\Big[\underbrace{\frac{1}{2} - \frac{1}{24} - 1}_{-5/24} + \frac{1}{3}\Big] = \boxed{\frac{\pi\sqrt{3}}{24} - \frac{5}{72}} = \boxed{\frac{3\sqrt{3}\pi - 5}{72}} \cong 0.1573$$

(b) (5 points) Use the comparison test to determine whether the following improper integral converges or diverges. Show your work.

$$\int_3^{\infty} \frac{dx}{x^4 + \cos^2 x}$$

For all x: $\cos^2 x \geq 0$, so $x^4 + \cos^2 x \geq x^4$

So, on $[3,\infty)$: $\frac{1}{x^4 + \cos^2 x} \leq \frac{1}{x^4}$

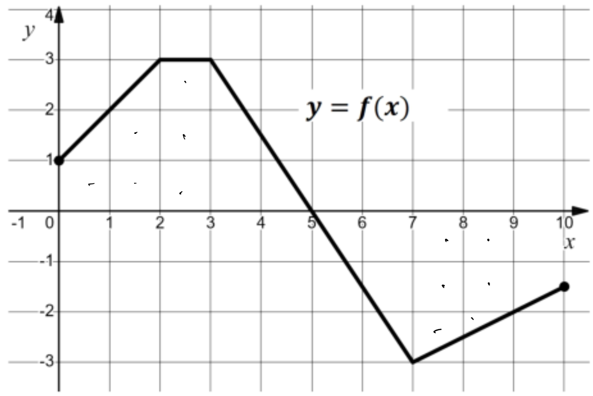
$$\Rightarrow \int_3^{\infty} \frac{1}{x^4+\cos^2 x}\,dx \leq \int_3^{\infty} \frac{1}{x^4}\,dx \quad \text{converges}$$

(we know that $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges for $p > 1$)

By the comparison test:

$$\boxed{\int_3^{\infty} \frac{1}{x^4+\cos^2 x}\,dx \text{ converges.}}$$

3. (10 points) Let $f(x)$ be the function whose graph, for $0 \leq x \leq 10$, is shown below:



Let $g(x) = \int_0^x f(t)\,dt$, and let $h(x) = \int_0^{x^2} f(t)\,dt$.

(a) (2 points) Evaluate $g(3)$ $= \int_0^3 f(t)\,dt$ = area over $[0,3]$ = 7

Answer: $g(3) =$ 7

(b) (3 points) Evaluate $g'(3)$. $= f(3)$ by FTC I

Answer: $g'(3) =$ 3

(c) (2 points) Evaluate $h(3)$. $= \int_0^9 f(t)\,dt$

$= \int_0^3 f(t)\,dt + \cancel{\int_3^7 f(t)\,dt}_{0} + \int_7^9 f(t)\,dt = 7 + 0 - 5$

Answer: $h(3) =$ 2

(d) (3 points) Evaluate $h'(3)$.

$h(x) = \int_0^{x^2} f(t)\,dt$

By FTC I + Chain:

$h'(x) = f(x^2) \cdot (x^2)' = f(x^2)(2x)$

$\therefore h'(3) = f(9)(6) = (-2)(6)$

Answer: $h'(3) =$ -12

4. (10 points) A particle is moving along a straight line with acceleration at t seconds given by:

$$a(t) = 2t - 6, \ \text{m/s}^2.$$

At time $t = 0$ seconds, its velocity is $v(0) = 8$ m/s.

(a) (5 points) Find the velocity $v(t)$ of the particle as a function of time t.

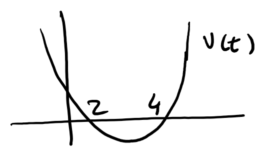
$$v(t) = \int a(t)\,dt = \int 2t - 6\,dt = t^2 - 6t + C$$

$$v(0) = 8 \Rightarrow C = 8$$

$$\boxed{v(t) = t^2 - 6t + 8}$$

(b) (5 points) What is the *total distance* traveled by the particle from time $t = 0$ to time $t = 4$ sec?

$v(t) = t^2 - 6t + 8$ is a concave-up parabola with roots where $t^2 - 6t + 8 = (t-4)(t-2) = 0$ i.e. at $t = 2$ and $t = 4$



On $[0,2]$: $v(t) \geq 0$, on $[2,4]$: $v(t) \leq 0$.

Total distance on $[0,4]$ $= \int_0^4 |v(t)|\,dt$

$$= \int_0^2 (t^2 - 6t + 8)\,dt + \int_2^4 -(t^2 - 6t + 8)\,dt$$

$$= \left(\tfrac{1}{3}t^3 - 3t^2 + 8t\right)\Big|_0^2 - \left[\tfrac{1}{3}t^3 - 3t^2 + 8t\right]\Big|_2^4$$

$$= \left(\tfrac{8}{3} - 12 + 16\right) - (0) - \left(\tfrac{4^3}{3} - 3(4)^2 + 8(4)\right) + \left(\tfrac{8}{3} - 12 + 16\right)$$

$$= \frac{20}{3} - \frac{16}{3} + \frac{20}{3} = \frac{24}{3} = \boxed{8 \text{ meters}}$$

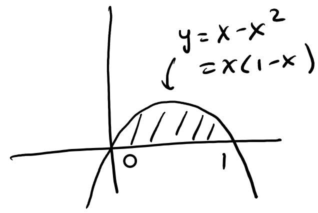
5. (a) (5 points) Find the area between the curve $y = x(1-x)$ and the x-axis over the interval $0 \leq x \leq 1$.

$$\text{area} = \int_0^1 (x - x^2)\, dx$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \boxed{\frac{1}{6}}.$$



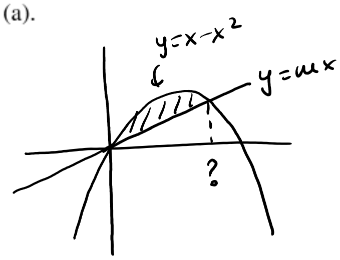
(b) (5 points) Find the constant m, with $0 < m < 1$, such that the area enclosed between the curve $y = x(1-x)$ and the line $y = mx$ is half of the total area from part (a).

Curves intersect when $mx = x - x^2$

$$x^2 + mx - x = 0$$

$$x(x + m - 1) = 0$$

$$x = 0 \quad \text{and} \quad x = 1 - m.$$



$$\text{Area between} = \int_0^{1-m} \left((x - x^2) - mx\right) dx = \frac{1}{2}\left(\frac{1}{6}\right) = \frac{1}{12}$$

$$\int_0^{1-m} \left((1-m)x - x^2\right) dx = \frac{1}{12}$$

$$\left((1-m)\frac{x^2}{2} - \frac{1}{3}x^3\right)\Big|_0^{(1-m)} = \frac{1}{12}$$

$$\frac{(1-m)^3}{2} - \frac{(1-m)^3}{3} = \frac{1}{12}$$

$$\frac{(1-m)^3}{6} = \frac{1}{12} \Rightarrow (1-m)^3 = \frac{1}{2}$$

$$\therefore 1 - m = \frac{1}{\sqrt[3]{2}} \Rightarrow \boxed{m = 1 - \frac{1}{\sqrt[3]{2}}} = \frac{\sqrt[3]{2} - 1}{\sqrt[3]{2}}$$

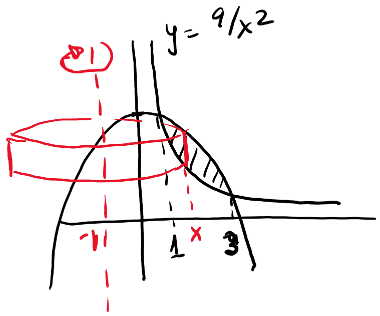
6. (10 points) Let R be the region with $x \geq 0$ enclosed between the graphs of

$$y = \frac{9}{x^2} \text{ and } y = 10 - x^2.$$

Find the volume of the solid of revolution obtained by rotating this region R about the line $x = -1$.

Intersection: $\frac{9}{x^2} = 10 - x^2 \Rightarrow 9 = 10x^2 - x^4 \Rightarrow x^4 - 10x^2 + 9 = 0$.

$\Rightarrow (x^2-9)(x^2-1) = 0 \Rightarrow \begin{cases} x^2 = 9 \Rightarrow x = \pm 3 \geq 0 \Rightarrow x = 3 \\ \text{or} \\ x^2 = 1 \Rightarrow x = \pm 1 \geq 0 \Rightarrow x = 1. \end{cases}$



Volume:

1) by Shells: $\int_1^3 2\pi r h \, dx$

OR $= \boxed{\int_1^3 2\pi (x+1)\left(10 - x^2 - 9/x^2\right) dx.}$

2) by Washers: $\int_1^9 \pi R^2 - \pi r^2 \, dy$

$= \boxed{\int_1^9 \pi \left(\sqrt{10-y}+1\right)^2 - \pi\left(\frac{3}{\sqrt{y}}+1\right)^2 dy.}$

(Either one $= \pi\left(\frac{152}{3} - 9\ln 9\right) \cong 97.05$)

7. (10 points) Compute the *arc length* of the curve $y = e^x$ over the interval $[0, 1]$.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = e^x$$

$$= \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

Substitution (either this one or incremental/multiple ones)

$$u = \sqrt{1 + e^{2x}} \Rightarrow u^2 = 1 + e^{2x} \Rightarrow e^{2x} = u^2 - 1 \Rightarrow \cancel{2} e^{2x} dx = \cancel{2} u\, du$$

$$\therefore dx = \frac{u}{u^2 - 1}\, du$$

$$L = \int_{x=0}^{x=1} \sqrt{1 + e^{2x}}\, dx = \int_{u=\sqrt{2}}^{u=\sqrt{1+e^2}} (u)\left(\frac{u}{u^2-1}\right) du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{(u^2-1)+1}{u^2-1}\, du$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} 1 + \frac{1}{(u-1)(u+1)}\, du$$

P.F.: $\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

$u = 1:\quad A = 1/2$

$u = -1\quad B = -1/2$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} 1 + \frac{1/2}{u-1} - \frac{1/2}{u+1}\, du$$

$$= \left[u + \frac{1}{2} \ln\left|\frac{u-1}{u+1}\right|\right]\Bigg|_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \boxed{\sqrt{1+e^2} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{1+e^2} - 1}{\sqrt{1+e^2} + 1} - \frac{1}{2} \ln \frac{\sqrt{2} - 1}{\sqrt{2} + 1}}$$

8. (10 points) A bucket of mass 4 kg is initially filled with water of mass 30 kg.

The bucket and water are lifted from the bottom of a well 10 meters deep up to the top of the well at a constant speed with a rope of mass 0.5 kg per meter.

The bucket leaks water at a constant rate and only 10 kg of water get to the top of the well. $\rightarrow (0,30)$ to $(10,10)$

slope $= -2$ kg/m.

How much work, in Joules, is done? You may use $g = 9.8$ m/s^2.

Let x = height above bottom of well, $0 \le x \le 10$.

$$\text{Work} = \int_0^{10} F_{bucket} + F_{water} + F_{rope} \, dx$$

$$= \int_0^{10} 9.8 \left(m_{bucket} + m_{water} + m_{rope}\right) dx$$

$$= 9.8 \int_0^{10} (4) + (30 - 2x) + 0.5(10 - x) \, dx$$

$$= 9.8 \int_0^{10} 39 - 2.5x \, dx$$

$$= 9.8 \left(39x - 1.25x^2\right)\Big|_0^{10}$$

$$= 9.8 (390 - 125)$$

$$= \boxed{2{,}597 \text{ Joules.}}$$

9. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = (y^2+1)\sqrt{4-x^2}, \; y(0) = 1$$

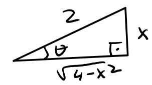
For full credit, write your answer in explicit form, $y = f(x)$.

Separate the variables:

$$\frac{1}{y^2+1}\frac{dy}{dx} = \sqrt{4-x^2}$$

$$\int \frac{1}{y^2+1}\,dy = \int \sqrt{4-x^2}\,dx$$

Trig Sub: $x = 2\sin\theta$
$dx = 2\cos\theta\, d\theta$



$$\int\sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$$
$$= \int 4\cos^2\theta\, d\theta = \int \cancel{4}^{2}\,\frac{1+\cos 2\theta}{\cancel{2}}\, d\theta$$
$$= 2\theta + \cancel{2}\,\frac{\sin 2\theta}{\cancel{2}} + C$$
$$= 2\arcsin\left(\frac{x}{2}\right) + 2\sin\theta\cos\theta$$
$$= 2\arcsin\left(\frac{x}{2}\right) + \cancel{2}\,\frac{x}{\cancel{2}}\,\frac{\sqrt{4-x^2}}{2}$$

Integrate each side:

$$\arctan y = 2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

Fix the constant: $x=0, y=1$:

$$\arctan 1 = \frac{\pi}{4} = 0 + 0 + C \Rightarrow C = \pi/4$$

Solve for y:

$$\boxed{y = \tan\left(2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + \frac{\pi}{4}\right)}$$

10. (10 points) A tank initially contains 90 liters of pure water.

A solution of salt in water that contains 0.2 kg of salt per liter enters the tank at a rate of 10 liters/min. In addition, another solution of salt in water that contains 0.1 kg of salt per liter enters the tank at a rate of 5 liters/min.

The solution is kept thoroughly mixed in the tank, and it drains from the tank at a rate of 15 liters/min.

How much salt will there be in the tank after t minutes?

Let $y = y(t)$ be the amount of salt in the tank, in kg, at t minutes.

We know: $y(0) = 0$

and: $\frac{dy}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$

$$= (0.2\,\text{kg/L})(10\,\text{L/min}) + (0.1\,\text{kg/L})(5\,\text{L/min}) - \left(\frac{y}{90}\,\text{kg/L}\right)(15\,\text{L/min})$$

$$= 2 + 0.5 - \frac{15}{90}y = 2.5 - \frac{1}{6}y$$

Solve the diff eq: $\frac{dy}{dt} = 2.5 - \frac{1}{6}y \quad , y(0) = 0.$

$$\frac{dy}{dt} = \frac{15-y}{6} = (y-15)\left(\frac{-1}{6}\right)$$

Separate vars.: $\int \frac{1}{y-15}\,dy = \int -\frac{1}{6}\,dt$

Integrate: $\ln|y-15| = -\frac{1}{6}t + C$

Solve for y: $|y-15| = e^{-t/6+C} = Ae^{-t/6} \qquad (A = e^C)$

$$y - 15 = Be^{-t/6} \qquad (B = \pm A)$$

$$y = 15 + Be^{-t/6}$$

Fix constant:

$y(0) = 0 \Rightarrow 0 = 15 + B \Rightarrow B = -15$

Answer: $\boxed{y(t) = 15 - 15e^{-t/6} = 15(1 - e^{-t/6})}$