1. Evaluate the following integrals. Show your work. Simplify and box your answers.

   (a) (5 points) $\displaystyle\int \frac{1}{x^4-x^2}\,dx$

Solution:
Factoring out the denominator and decomposing into Partial Fractions:

$$\begin{aligned}\frac{1}{x^4-x^2} &= \frac{1}{(x)^2(x-1)(x+1)}\\ &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}.\end{aligned}$$

Recombining the fractions and setting the numerators equal:

$$1 = A(x)(x-1)(x+1)+B(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$$

Solving, we get $B=-1$, $C=1/2$, $D=-1/2$, and $A=0$.
Hence:

$$\int \frac{1}{x^4-x^2}\,dx = \int \frac{-1}{x^2}+\frac{1/2}{x-1}+\frac{-1/2}{x+1}\,dx = \boxed{\frac{1}{x}+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C}$$

   (b) (5 points) $\displaystyle\int \sin^3(t)\,dt$

Solution:

$$\int \sin^3(t)\,dt = \int \sin^2(t)\sin(t)\,dt = \int \left(1-\cos^2(t)\right)\sin(t)\,dt.$$

Substituting $u=\cos(t)$, $du=-\sin(t)\,dt$

$$\begin{aligned}\int \sin^3(t)\,dt &= \int (1-u^2)(-1)\,du = \int (u^2-1)\,du\\ &= \frac{1}{3}u^3-u+C\\ &= \boxed{\frac{1}{3}\cos^3(t)-\cos(t)+C}\end{aligned}$$

2. Evaluate the following integrals. Show your work. Simplify and box your answers.

   (a) (5 points) $\displaystyle\int_{-1}^{0} \frac{x}{x^2+2x+2}\,dx$

Solution: We use the substitution $u = x+1$.

$$\int_{-1}^{0} \frac{x}{x^2+2x+2}dx = \int_{-1}^{0} \frac{x}{(x+1)^2+1}dx$$

$$= \int_{0}^{1} \frac{u-1}{u^2+1}du$$

$$= \int_{0}^{1} \frac{u}{u^2+1}du - \int_{0}^{1} \frac{1}{u^2+1}du$$

For the first integral we use the substitution $v = u^2$, $dv = 2udu$

$$\int_{0}^{1} \frac{u}{1+u^2}du - \int_{0}^{1} \frac{1}{1+u^2}du = \int_{0}^{1} \frac{1}{2(1+v)}dv - \int_{0}^{1} \frac{1}{1+u^2}du$$

$$= (1/2)\ln(1+v)\Big|_0^1 - \arctan u\Big|_0^1$$

$$= \boxed{(1/2)\ln 2 - \pi/4} \approx -0.438825.$$

Solution: Alternatively, we can use trig substitution: $x+1 = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$

$$\int_{-1}^{0} \frac{x}{x^2+2x+2}dx = \int_{-1}^{0} \frac{x}{(x+1)^2+1}dx$$

$$= \int_{0}^{\pi/4} \frac{\tan(\theta)-1}{1+\tan^2(\theta)}\sec^2(\theta)d\theta$$

$$= \int_{0}^{\pi/4} \tan(\theta) - 1\,d\theta$$

$$= (\ln|\sec(\theta)| - \theta)\Big|_0^{\pi/4} = \boxed{\ln(\sqrt{2}) - \pi/4}$$

(b) (5 points) $\int e^{2x} \sin x \, dx$

Solution:
We use integration by parts.

$$u = e^{2x}, \qquad dv = \sin x \, dx,$$
$$du = 2e^{2x} dx, \qquad v = -\cos x.$$

$$\int e^{2x} \sin x \, dx = -\cos x \, e^{2x} + 2 \int \cos x \, e^{2x} \, dx. \tag{1}$$

We use integration by parts again.

$$u = e^{2x}, \qquad dv = \cos x \, dx,$$
$$du = 2e^{2x} dx, \qquad v = \sin x.$$

$$\int e^{2x} \cos x dx = \sin x \, e^{2x} - 2 \int \sin x \, e^{2x} dx.$$

We substitute this into (1).

$$\int e^{2x} \sin x dx = -\cos x \, e^{2x} + 2 \left(\sin x \, e^{2x} - 2 \int \sin x e^{2x} \, dx \right)$$

$$5 \int e^{2x} \sin x dx = -\cos x \, e^{2x} + 2 \sin x \, e^{2x} + C.$$

$$\int e^{2x} \sin x \, dx = \boxed{-(1/5) \cos x \, e^{2x} + (2/5) \sin x \, e^{2x} + C}$$

3. (10 points) Find the arc length of the curve $y = \frac{1}{16}e^{2x} + e^{-2x}$ for $0 \le x \le 1$.

Solution: Note that

$$y' = \frac{1}{8}e^{2x} - 2e^{-2x}$$

and

$$(y')^2 = \frac{1}{64}e^{4x} - \frac{1}{2} + 4e^{-4x}$$

So

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+(y')^2}dx = \sqrt{1+\left(\frac{1}{8}e^{2x} - 2e^{-2x}\right)^2}dx \\
&= \int_0^1 \sqrt{\left(\frac{1}{8}e^{2x} + 2e^{-2x}\right)^2}dx \\
&= \int_0^1 \left(\frac{1}{8}e^{2x} + 2e^{-2x}\right)dx \\
&= \left(\frac{1}{16}e^{2x} - e^{-2x}\right)\Big|_0^1 \\
&= \boxed{\frac{1}{16}e^2 - e^{-2} + \frac{15}{16}}.
\end{aligned}$$

4. (10 points) The **position** of a particle at time t is given by $s(t) = t^3 - 2t^2 - 4t + 20$.
   Find the total distance traveled by the particle from $t = 0$ to $t = 3$.

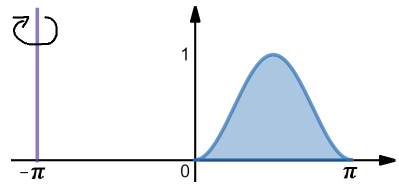
Solution: We compute the velocity of this particle:

$$v(t) = s'(t) = 3t^2 - 4t - 4 = (3t+2)(t-2),$$

By factoring or by the quadratic formula we determine that it's zero at $t = 2$ and $t = -2/3$ and over the interval $[0,3]$ we determine that: $v(t) < 0$ for $0 \le t < 2$ and $v(t) > 0$ for $t > 2$.

$$\begin{aligned}
\text{total distance traveled} &= \int_0^3 |v(t)|\,dt \\
&= \int_0^2 -(3t^2 - 4t - 4)dt + \int_2^3 (3t^2 - 4t - 4)dt \\
&= -(t^3 - 2t^2 - 4t)\Big|_0^2 + (t^3 - 2t^2 - 4t)\Big|_2^3 \\
&= -(8 - 8 - 8) + [(27 - 18 - 12) - (8 - 8 - 8)] \\
&= 8 + 5 \\
&= \boxed{13}.
\end{aligned}$$

5. The region below $y = \sin^2 x$, above the x-axis, between $x = 0$ and $x = \pi$ is rotated about the line $x = -\pi$ to generate a solid of revolution.



(a) (5 points) Write down an integral equal to the volume of the resulting solid of revolution. Do not compute the integral yet.

Solution:
Looking at the situation, it's by far best to set up the integral in x, which in this case means using the "shells" method. The volume is equal to

$$V = \int_0^\pi 2\pi R(x)h(x)dx$$

$$= \boxed{\int_0^\pi 2\pi(x+\pi)\sin^2(x)\,dx}$$

(b) (5 points) Compute the integral to determine the volume of the solid of revolution. Give your answer in exact form or as a decimal number with at least four significant digits.

Solution:

$$V = \int_0^\pi 2\pi(x+\pi)\sin^2 x dx = 2\pi\left(\int_0^\pi x\sin^2 x dx + \pi\int_0^\pi \sin^2 x dx\right)$$

For the second integral, recall that $\sin^2 x = (1/2)(1-\cos(2x))$.

$$\int_0^\pi \sin^2 x dx = \int_0^\pi (1/2)(1-\cos(2x))dx = (1/2)\left(x-(1/2)\sin(2x)\right)\Big|_0^\pi = \pi/2.$$

For the first integral, we will use integration by parts.

$$u = x, \qquad dv = \sin^2 x\,dx,$$
$$du = dx, \qquad v = (1/2)x - (1/4)\sin(2x)$$

$$\int_0^\pi x\sin^2 x dx = x((1/2)x-(1/4)\sin(2x))\Big|_0^\pi - \int_0^\pi ((1/2)x-(1/4)\sin(2x))dx$$
$$= \pi^2/2 - \left((1/4)x^2+(1/8)\cos(2x)\right)\Big|_0^\pi = \pi^2/2-(1/4)\pi^2 = \pi^2/4.$$

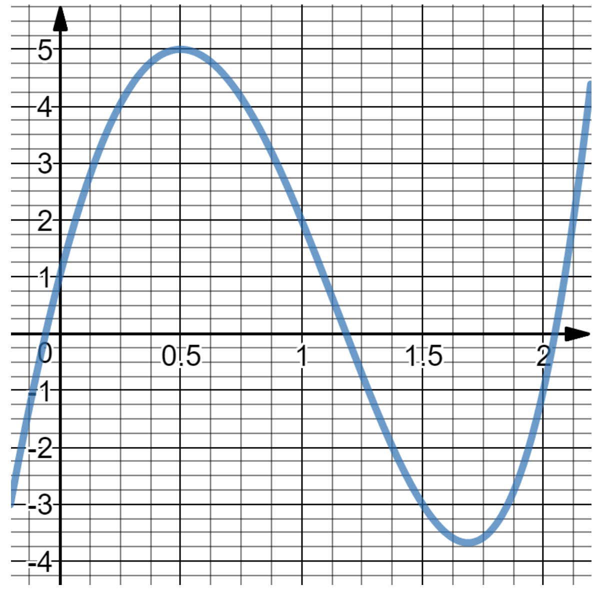
Altogether, we obtain $V = 2\pi\left(\pi^2/4+\pi(\pi/2)\right) = \boxed{3\pi^3/2} \approx 46.5094$.

Alternatively, we could compute the original integral directly via integration by parts with

$$u = x+\pi, \qquad dv = \sin^2 x dx,$$
$$du = dx, \qquad v = (1/2)x-(1/4)\sin(2x)$$

6. (10 points) The graph of a function $f(t)$ is shown. Use it to answer the following questions.

(a) (5 points) Approximate the **average value** of this function over the interval $[0,2]$ using Simpson's Rule with $n = 4$ subintervals.



Solution:

$$f_{ave} = \frac{1}{2}\int_0^2 f(t)\,dt$$

$$\approx \frac{1}{2}\left(\frac{0.5}{3}\left(f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)\right)\right)$$

$$= \frac{1}{2}\left(\frac{0.5}{3}\left(1+4\cdot 5+2\cdot 2+4\cdot(-3)+(-1)\right)\right) = \frac{1}{2}\left(\frac{0.5}{3}(12)\right) = \boxed{1}$$

(b) (5 points) Define a new function: $g(x) = \int_x^{2x} f(t)dt$, where f is the function in the graph above. Compute $g'(0.5)$

Solution:

$$g(x) = \int_x^{2x} f(t)dt = -\int_0^x f(t)dt + \int_0^{2x} f(t)dt$$

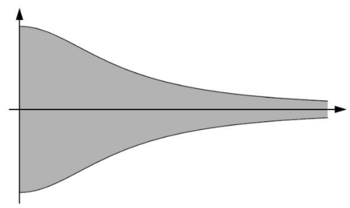
Applying the Fundamental Theorem of Calculus (and Chain Rule), we get:

$$g'(x) = -\frac{d}{dx}\left(\int_0^x f(t)dt\right) + \frac{d}{dx}\left(\int_0^{2x} f(t)dt\right) = -f(x) + f(2x)\cdot 2$$

Evaluating at $x = 0.5$,

$$g'(0.5) = -f(0.5) + f(1)\cdot 2 = -5 + 2\cdot 2 = \boxed{-1}$$

7. Consider the region bounded above by the graph of $y = \frac{1}{1+x^2}$, bounded below by the graph of $y = \frac{-1}{1+x^2}$, bounded on the left by the y axis, and with no bound on the right.



(a) (5 points) Find the area of this region. If it's finite, compute its value. If it's infinite, show why. Show all steps and limit computations in either case.

Solution:

$$A = \int_0^\infty \frac{1}{1+x^2} - \frac{-1}{1+x^2} dx = 2\int_0^\infty \frac{1}{1+x^2} dx$$

Writing the improper integral as a limit:

$$A = 2\lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx$$

$$= 2\lim_{t\to\infty} \left(\arctan(x)\Big|_0^t \right)$$

$$= 2\lim_{t\to\infty} (\arctan(t) - 0)$$

$$= 2\frac{\pi}{2} = \boxed{\pi}$$

(b) (5 points) Find the $\bar{x}$ coordinate of the center of mass of this region. If it's finite, compute its value. If it's infinite, show why. Show all steps and limit computations in either case.

Solution:

$$\bar{x} = \frac{1}{A}\int_0^\infty x \cdot \frac{1}{1+x^2} - x \cdot \frac{-1}{1+x^2} dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2} dx$$

Substituting $u = x^2 + 1$, $du = 2xdx$:

$$\bar{x} = \frac{2}{\pi}\int_1^\infty \frac{1/2}{u} du$$

$$= \frac{1}{\pi}\lim_{t\to\infty} \left(\ln|u|\Big|_1^t \right)$$

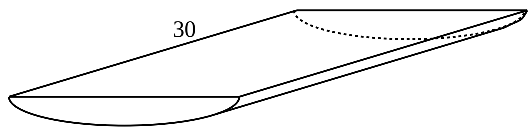
$$= \frac{1}{\pi}\lim_{t\to\infty} (\ln(t))$$

$$= \boxed{\infty}$$

8. (10 points) A part of an elevated freeway is 30 meters long and has half elliptical cross sections, as shown below. The width at the top of the cross section (the horizontal axis of the ellipse) is 4 meters. The height in the middle of the cross section (half of the vertical axis of the ellipse) is 1 meter. The lowest point of the section is 5 meters above the ground.

   The structure is made of concrete which was pumped from the ground level up, in liquid state. The mass density of concrete is 2,400 kg per cubic meter. The gravitational acceleration is 9.8 m/sec^2.

   SET UP (do NOT evaluate) an integral equal to the WORK that was done to pump the concrete up to make the section of the structure. The work done to make the support column should be ignored.



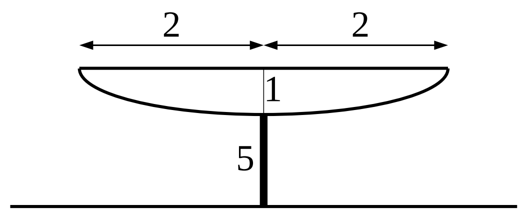


Figure 1: *The upper part of the structure is depicted on the left. The cross section is depicted on the right. Recall that the equation of an ellipse* ***centered at the origin*** *is* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$*, where a and b are half of the horizontal and vertical axes of the ellipse, respectively.*

Solution:

We take the zero level to be at the top of the structure.
Then the equation for the ellipse is $(x/2)^2+y^2=1$.
Hence, $x=2\sqrt{1-y^2}$
With our choice of coordinates, y denotes the (positive) distance from the upper level of the structure to the horizontal layer of concrete, and concrete section of the structure corresponds to the interval $[0,1]$. The i^{th} horizontal layer of concrete is very thin and rectangular, with volume

$$30\times(2x_i)\times\Delta x$$

The force needed to lift it is $2400\times 9.8\times(30\times 2x_i\times\Delta x)$ N.
The distance this layer must be lifted is $6-y_i$ meters.

The total work, in joules, is

$$\boxed{\int_0^1 2400\cdot 9.8\cdot 30(6-y)2\cdot 2\sqrt{1-y^2}dy = 2,822,400\int_0^1(6-y)\sqrt{1-y^2}dy}$$

9. (10 points) Find the solution of the initial value problem

$$y' = (y^2+4)^{3/2} x \ln x \quad \text{with } y(1) = 2.$$

Leave your answer as an implicit equation involving the variables y and x.

Solution: Since $(y^2+4)^{-3/2}y' = x\ln x$,

$$\int (y^2+4)^{-3/2} dy = \int x \ln x \, dx.$$

Using $y = 2\tan\theta$,

$$\begin{aligned}
\int (y^2+4)^{-3/2} dy &= \int (2\sec\theta)^{-3} \cdot 2\sec^2\theta d\theta \\
&= \frac{1}{4}\int \cos\theta d\theta = \frac{1}{4}\sin\theta + c \\
&= \frac{1}{4}\frac{y}{\sqrt{y^2+4}} + c.
\end{aligned}$$

On the other hand, using integration by parts,

$$\begin{aligned}
\int x \ln x dx &= \frac{1}{2}\int \ln x \, d(x^2) \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c.
\end{aligned}$$

Thus we have

$$\frac{1}{4}\frac{y}{\sqrt{y^2+4}} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c.$$

Or, equivalently,

$$\frac{y}{\sqrt{y^2+4}} = 2x^2 \ln x - x^2 + c_1.$$

Since $y(1) = 2$, $\frac{2}{\sqrt{8}} = 0 - 1 + c_1$ and so $c_1 = \frac{1}{\sqrt{2}} + 1$. Hence the solution y satisfies

$$\boxed{\frac{y}{\sqrt{y^2+4}} = 2x^2 \ln x - x^2 + \frac{1}{\sqrt{2}} + 1.}$$

10. At this time, the population of Mountainstan is 100 million people. The natural growth rate (births minus deaths) is 2% of population per year. Each year 1 million people emigrate from Mountainstan to other countries looking for better economic opportunities. Assume no people move from other countries to Mountainstan.

 (a) (4 points) Let $y = y(t)$ be the population of Mountainstan, in millions, t years from now.
 Write down a differential equation for y. Do NOT solve the equation yet.

 Solution: y - population size in millions, t - time in years

$$\frac{dy}{dt} = 0.02y - 1$$

 (b) (6 points) Assuming that these trends will be stable for a long time, determine how many years from now the population will reach 150 million. Give your answer as a decimal number with at least 4 significant digits.

 Solution:

$$\frac{dy}{0.02y-1} = dt$$

$$\int \frac{dy}{0.02y-1} = \int dt$$

$$\frac{1}{0.02}\ln|0.02y-1| = t + c_1$$

$$\ln|0.02y-1| = 0.02t + c_2$$

$$0.02y - 1 = \pm e^{0.02t+c_2} = c_3 e^{0.02t}$$

 Since $y(0) = 100$, we have

$$0.02 \cdot 100 - 1 = c_3 e^{0.02\cdot 0} = c_3$$

 so $c_3 = 1$.
 We obtain

$$0.02y - 1 = e^{0.02t}$$

$$y = 50(e^{0.02t} + 1)$$

 We are looking for t such that $y(t) = 150$.

$$150 = 50(e^{0.02t} + 1)$$

$$t = \frac{1}{0.02}\ln(2) = 50\ln 2 \approx \boxed{34.6574}$$