Solutions

1. Evaluate the following integrals. Show clear and complete work.

(a) (6 points) $\int \tan^2(5x)\sec^4(5x)\,dx$

$$= \int \tan^2(5x)\sec^2(5x)\sec^2(5x)\,dx$$

$$= \int \tan^2(5x)\left(1+\tan^2(5x)\right)\underbrace{\sec^2(5x)\,dx}_{\frac{1}{5}du}$$

$$\boxed{\begin{aligned} u &= \tan(5x) \\ du &= 5\sec^2(5x)\,dx \end{aligned}}$$

(or: $u = 5x$ then $v = \tan(u)$)

$$= \frac{1}{5}\int u^2(1+u^2)\,du$$

$$= \frac{1}{5}\int u^2 + u^4\,du$$

$$= \frac{1}{5}\left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right] + C$$

Answer: $\frac{1}{15}(\tan(5x))^3 + \frac{1}{25}(\tan(5x))^5 + C$

(b) (6 points) $\int_1^e \frac{3\ln(x)}{x\sqrt{4+(\ln x)^2}}\,dx$

1) $\boxed{u = \ln x,\ du = \frac{1}{x}dx}$

$$= \int_{\ln 1 = 0}^{\ln e = 1} \frac{3u}{\sqrt{4+u^2}}\,du$$

2) $w = 4+u^2$ (OR $v = \sqrt{4+u^2}$, $v^2 = 4+u^2$, $\cancel{2}v\,dv = \cancel{2}u\,du$)

$\frac{1}{2}dw = \cancel{2}u\,du$

$$= \int_{w=4}^{w=5} \frac{3}{\sqrt{w}}\frac{1}{2}dw$$

$$= \frac{3}{2}\int_4^5 w^{-1/2}\,dw$$

$$= \frac{3}{\cancel{2}}\left(\cancel{2}\sqrt{w}\right)\Big|_4^5$$

$$= 3\sqrt{5} - 6$$

Answer: $3\sqrt{5} - 6 = 3(\sqrt{5} - 2)$

2. (6 points) Use the Comparison Test to determine if the following integral converges or diverges.

$$\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$$

This improper integral (check one box):

- ☑ Converges
- ☐ Diverges

To indicate the specific comparison you used to reach your conclusion, select one correct and relevant fact from each of the two columns below. No need to justify further.

BECAUSE, for all $x \geq 1$:

- ☐ $\frac{1}{\sqrt{x}+x^2} < \frac{1}{\sqrt{x}}$ (true but not useful for comparison)
- ☑ $\frac{1}{\sqrt{x}+x^2} < \frac{1}{x^2}$
- ☐ $\frac{1}{\sqrt{x}+x^2} > \frac{1}{\sqrt{x}}$
- ☐ $\frac{1}{\sqrt{x}+x^2} > \frac{1}{x^2}$

these are FALSE

AND:

- ☐ $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converges FALSE
- ☐ $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (true but not useful)
- ☑ $\int_1^\infty \frac{1}{x^2}\,dx$ converges
- ☐ $\int_1^\infty \frac{1}{x^2}\,dx$ diverges FALSE

3. (10 points) A car is traveling at a constant velocity of 27 m/s when its driver notices a road block ahead. The driver takes 0.5 seconds to react and start braking, and then brakes with a deceleration

$$a(t) = -3t^2 \ m/s^2,$$

where t is the number of seconds since it started braking, until the car comes to a full stop.

How many meters will the car travel from the first moment when the driver noticed the road block until the car comes to a full stop?

1) First 0.5 sec at 27 m/s $\Rightarrow$ distance $d_1 = (0.5)(27) = 13.5$ meters.

2) Once the driver starts braking:

$$\left.\begin{array}{l}\text{initial velocity } v(0) = 27 \text{ m/s} \\ a(t) = -3t^2 \Rightarrow v(t) = -t^3 + C\end{array}\right\} \Rightarrow v(t) = -t^3 + 27$$

car comes to a stop when $v(t) = -t^3 + 27 = 0$

$\Leftrightarrow t^3 = 27$

$\Leftrightarrow t = 3$ sec (after it started braking)

distance traveled while braking:

$$d_2 = \int_0^3 v(t)\,dt = \int_0^3 (27 - t^3)\,dt$$

$$= 27t - \frac{1}{4}t^4 \Big|_0^3$$

$$= 27(3) - \frac{1}{4}(81) = 81\left(1 - 1/4\right) = \frac{243}{4} = 60.75 \text{ meters}$$

Total: $d_1 + d_2 = 13.5 + 60.75 = 74.25$

Answer: 74.25 meters

4. (10 points) Let $\mathcal{R}$ be the shaded region shown. It is bounded above by curve:

$$y = 1 + \sqrt{1-x^2}, \quad \text{for } 0 \le x \le 1,$$

and it is bounded below by the lines $y = 1/2$ and $y = x$.

Set up, but do not evaluate, an integral (or a sum of integrals) equal to the volume of the solid of revolution obtained by revolving $\mathcal{R}$ around **the vertical line** $x = -1$.

Indicate which method you are using, and show some work. Box your final answer.

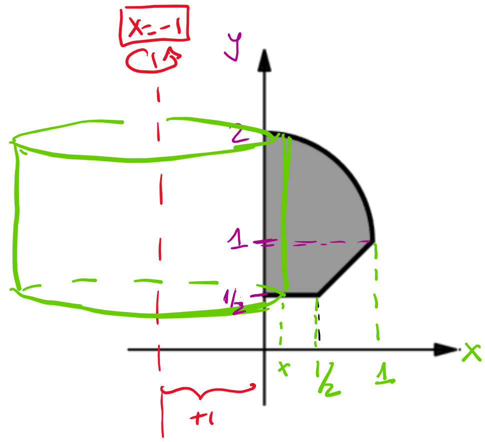
$y = 1 + \sqrt{1-x^2} \Rightarrow \sqrt{1-x^2} = y - 1$

$1 - x^2 = (y-1)^2$

$x^2 = 1 - (y-1)^2$

$x = \sqrt{1-(y-1)^2}$

$= \sqrt{-y^2+2y}$



Method I: $\boxed{\text{Shells in } x}$:

$$\text{Vol} = \int_0^{1/2} 2\pi(\text{radius})(\text{height})\,dx + \int_{1/2}^{1} 2\pi(\text{radius})(\text{height})\,dx$$

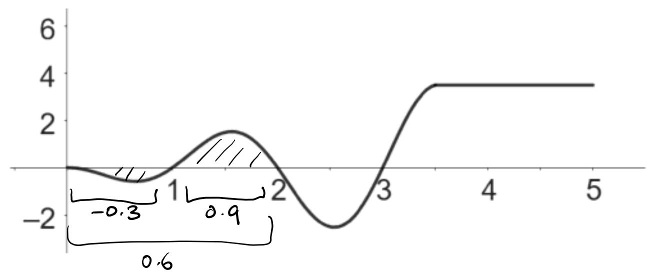
$$\boxed{= \int_0^{1/2} 2\pi (x+1)\left(1+\sqrt{1-x^2} - \tfrac{1}{2}\right)dx + \int_{1/2}^{1} 2\pi (x+1)\left(1+\sqrt{1-x^2} - x\right)dx}$$

Method II: $\boxed{\text{Washers in } y}$

$$\text{Vol} = \int_{1/2}^{1} \pi R^2(y) - \pi r^2(y)\,dy + \int_1^2 \pi R^2(y) - \pi r^2(y)\,dy$$

$$= \int_{1/2}^{1} \pi (y+1)^2 - \pi (1)^2\,dy + \int_1^2 \pi\left(\sqrt{-y^2+2y}+1\right)^2 - \pi(1)^2\,dy$$

5. (10 points) Below is shown the graph of a function $y = f(t)$ defined on $0 \le x \le 5$.



In addition, you know that: $\int_0^2 f(x)\,dx = 0.6$, $\int_1^2 f(x)\,dx = 0.9$, and $\int_2^4 f(x)\,dx = 1.3$.

(a) Calculate $\int_0^2 |f(x)|\,dx$.

$= \left|\int_0^1 f(x)dx\right| + \int_1^2 f(x)\,dx$

$= |-0.3| + 0.9$

Answer: $\int_0^2 |f(x)|\,dx = \boxed{1.2}$

(b) Calculate $\int_0^2 x f(x^2)\,dx$. $u = x^2$, $\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int_0^4 f(u)\,du$

$= \frac{1}{2}(0.6 + 1.3) = \frac{1.9}{2}$

Answer: $\int_0^2 x f(x^2)\,dx = \boxed{0.95}$

(c) Let $F(x) = \int_1^x f(t)\,dt$, with f as in the graph above.

i. Circle the correct answer (no need to justify): $F(0)$ is

POSITIVE NEGATIVE ZERO

ii. Circle the correct answer (no need to justify): $F'(1.5)$ is

POSITIVE NEGATIVE ZERO

6. (10 points) The electric force F (in Newtons) acting on a charged particle B as a result of the presence of a second charged particle A is given by Coulomb's Law:

$$F = k\frac{q_A q_B}{r^2}, \quad = k\,\frac{(1)(-1)}{r^2}$$

where $k = 9 \times 10^9$ is a constant, q_A and q_B are the charges of the particles A and B, in Coulombs, and r is the distance (in meters) between the particles.

Assume that two particles A and B have charges $q_A = 1$ and $q_B = -1$ Coulomb. (The force acting on them is negative, indicating that the particles are attracting each other.) Assume that particle A is kept fixed, while particle B is being moved away from A by applying a force of equal magnitude and opposite sign to the force by which A attracts B.

(a) Find the work done to move particle B from an initial position of 1 meter away from particle A, to a position 2 meters away from A.

$$\int_1^2 (\text{force})\,dr = \int_1^2 k\frac{1}{r^2}\,dr = k\int_1^2 \frac{1}{r^2}\,dr$$

$$= k\left(\frac{-1}{r}\right)\Big|_{r=1}^{r=2}$$

$$= k\left[-\frac{1}{2}+1\right]$$

$$= \frac{1}{2}k = \frac{1}{2}(9\times10^9)$$

Answer: work= $(4.5)\times 10^9$ Joules

(b) Find the work done to move particle B from an initial position of 1 meter away from A, to an infinite distance away from particle A. If that work is infinite, say so and justify your claim.

$$\int_1^\infty (\text{force})\,dr = \int_1^\infty k\frac{1}{r^2}\,dr = \lim_{b\to\infty}\left(\frac{-k}{r}\right)\Big|_1^b$$

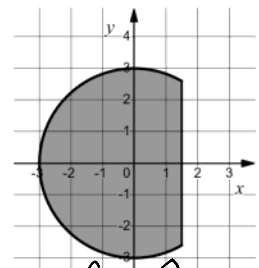
$$= \lim_{b\to\infty}\left(-\frac{k}{b}+\frac{k}{1}\right)$$

$$= 0 + k = k$$

Answer: work= 9×10^9 Joules

7. Let R be the portion of the disk of radius 3 centered at the origin for $-3 \le x \le 1.5$, as shown below.

(a) (8 points) Compute the area of this region R.



circle $x^2+y^2=9 \Rightarrow y = \pm\sqrt{9-x^2}$

OR $\dfrac{\pi(3)^2}{2} + \int_0^{1.5} 2\sqrt{9-x^2}\,dx$

$$\text{Area} = \int_{-3}^{1.5} \sqrt{9-x^2} - (-\sqrt{9-x^2})\,dx$$

$$= 2\int_{-3}^{1.5} \sqrt{9-x^2}\,dx \longrightarrow \boxed{x = 3\sin\theta,\ dx = 3\cos\theta\, d\theta}$$

Trig Sub

$\begin{bmatrix} x=-3: & \theta = \arcsin(-1) = -\pi/2 \\ x = 1.5 & \theta = \arcsin(1/2) = \pi/6 \end{bmatrix}$

$$= 2\int_{-\pi/2}^{\pi/6} \underbrace{3\cos\theta}\ \underbrace{3\cos\theta\, d\theta}$$

$$= \cancel{2}\int_{-\pi/2}^{\pi/6} 9\,\frac{1+\cos(2\theta)}{\cancel{2}}\,d\theta = 9\left[\theta + \frac{\sin 2\theta}{2}\right]\Bigg|_{-\pi/2}^{\pi/6}$$

$$= 9\left(\pi/6 + \pi/2\right) + \frac{9}{2}\left(\sin\pi/3 - \underbrace{\cancel{\sin}(-\pi)}_{0}\right)$$

$$= 9\,\frac{4\pi}{6} + \frac{9}{2}\,\frac{\sqrt{3}}{2}$$

$$= 6\pi + \frac{9\sqrt{3}}{4} \cong 22.7466$$

AREA = $\dfrac{24\pi + 9\sqrt{3}}{4}$ $(\cong 22.7466)$

(b) (8 points) Find the coordinates $(\bar{x}, \bar{y})$ for the center of mass of a thin lamina of uniform area density $\rho = 1$ occupying this region R. Round to nearest two decimal digits.

$\bar{y} = 0$ (by symmetry)

$$\bar{x} = \frac{1}{A}\int_{-3}^{1.5} x\sqrt{9-x^2} - x(-\sqrt{9-x^2})\,dx = \frac{1}{A}\int_{-3}^{1.5 = 3/2} 2x\sqrt{9-x^2}\,dx$$

$$= \frac{1}{A}\int_0^{27/4} \sqrt{u}\,(-1)\,du \quad \boxed{\begin{matrix}\text{with } u = 9-x^2 \\ -du = +2x\,dx\end{matrix}}$$

& bounds: $x = -3 \Rightarrow u = 9-9 = 0$
$x = 3/2 \Rightarrow u = 9 - \frac{9}{4} = \frac{27}{4}$

$$= \frac{1}{A}\,\frac{-2}{3}\,u^{3/2}\Big|_0^{27/4}$$

$$= \frac{4}{24\pi + 9\sqrt{3}}\left(\frac{-2}{3}\right)\left(\frac{27}{4}\right)^{3/2}$$

$$\cong -0.51398\ldots$$

$(\bar{x}, \bar{y}) =$ $(-0.51, 0)$

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = xe^{2x+y}, \quad y(0) = 1$$

For full credit, show all steps and write your answer in explicit form, $y = f(x)$.

Separate the variables: $\frac{dy}{dx} = x\,e^{2x}e^{y}$

$$\frac{1}{e^y}\frac{dy}{dx} = x\,e^{2x}$$

and integrate: $\int e^{-y}\,dy = \int x\,e^{2x}\,dx$ ← IBP $u = x$, $dv = e^{2x}dx$, $du = dx$, $v = \frac{1}{2}e^{2x}$

$$-e^{-y} = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$$

$$-e^{-y} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$e^{-y} = \frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + C_1$$

Using $y(0)=1$ and solving for C:

$$e^{-1} = \frac{1}{4} - \cancel{\frac{1}{2}(0)} + C_1$$

$$C_1 = \frac{1}{e} - \frac{1}{4}$$

So: $e^{-y} = \underbrace{\frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{e}}\, - \frac{1}{4}$

$$-y = \ln\left(\qquad \right)$$

$$y = -\ln\left(\frac{1}{4}e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{e} - \frac{1}{4}\right)$$

Answer: $y = -\ln\left(1/4\, e^{2x} - 1/2\, x e^{2x} + 1/e - 1/4\right)$

9. A population of yeast cells in certain container is found to satisfy the differential equation:

$$\frac{dP}{dt} = 3P - \frac{P^2}{1000} = \frac{3000P - P^2}{1000}$$

where $P(t)$ is the number of yeast cells in the container after t hours.

(a) (4 points) What are all the equilibrium solutions ($P =$ a constant) for this differential equation?

$P =$ ct when $\frac{dP}{dt} = 0$ at all t

so $0 = 3P - \frac{P^2}{1000} = P\left(3 - \frac{P}{1000}\right)$

True when $P = 0$ or $3 = \frac{P}{1000}$

$P = 3000$

Answer: $P = 0$ & $P = 3000$.

(b) (12 points) Solve the given differential equation and determine the number of yeast cells after 2 hours if the population started with 1000 cells. *Show clear and complete work; use additional pages if needed. Give your answer in exact form.*

$$\frac{dP}{dt} = \frac{3000P - P^2}{1000}$$

Separate the variables: $\int \frac{1000}{P(3000-P)} dP = \int 1 \, dt$

Partial Fractions: $\frac{1000}{P(3000-P)} = \frac{A}{P} + \frac{B}{3000-P}$

$1000 = A(3000-P) + BP \longrightarrow$ $P=0$: $P=3000$: $\boxed{A = 1/3 \quad B = 1/3}$

Integrating: $\int \frac{1/3}{P} dP + \int \frac{1/3}{3000-P} dP = t + C$

$\frac{1}{3}\left(\ln|P| - \ln|3000-P|\right) = t + C$

$\ln\left|\frac{P}{3000-P}\right| = 3t + C_1$

$t=0$: $P(0) = 1000$: $\ln\left|\frac{1000}{3000-1000}\right| = C_1 \Rightarrow C_1 = \ln(1/2)$

After $t = 2$ hrs: $\ln\left|\frac{P(2)}{3000-P(2)}\right| = 6 + \ln(1/2) \Rightarrow \Bigg|\underbrace{\frac{P(2)}{3000-P(2)}}_{>0.}\Bigg| = e^6 \cdot \cancel{e}^{\ln(1/2)} = \frac{1}{2}e^6$

$\Rightarrow P(2) = \frac{1}{2}e^6(3000 - P(2))$

$\Rightarrow 2P(2) = 3000e^6 - e^6 P(2)$

$\Rightarrow P(2)(2 + e^6) = 3000e^6 \Rightarrow \boxed{P(2) = \frac{3000e^6}{2+e^6}}$

Answer (rounded to nearest whole number of cells) = 2985