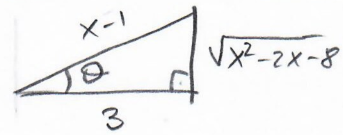
1. (14 total points) Evaluate the following integrals. Show your work.

(a) (7 points) $\int \frac{x}{\sqrt{x^2-2x-8}}\,dx = \int \frac{x}{\sqrt{(x-1)^2-9}}\,dx$

Trig Sub:

$\boxed{\begin{aligned} x-1 &= 3\sec\theta \\ dx &= 3\tan\theta\sec\theta\,d\theta \end{aligned}}$



$\sec\theta = \frac{x-1}{3} = \frac{hyp}{adj}$

$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{x^2-2x-8}}{3}$

$$= \int \frac{3\sec\theta+1}{\sqrt{9\sec^2\theta-9}}\,3\tan\theta\sec\theta\,d\theta$$

$$= \int \frac{3\sec\theta+1}{\cancel{3}\cancel{\tan\theta}}\,\cancel{3}\cancel{\tan\theta}\sec\theta\,d\theta$$

$$= \int 3\sec^2\theta + \sec\theta\,d\theta$$

$$= 3\tan\theta + \ln|\sec\theta+\tan\theta| + C$$

$$= \cancel{3}\frac{\sqrt{x^2-2x-8}}{\cancel{3}} + \ln\left|\frac{x-1}{3} + \frac{\sqrt{x^2-2x-8}}{3}\right| + C$$

rename + C

$$= \sqrt{x^2-2x-8} + \ln|x-1+\sqrt{x^2-2x-8}|\boxed{-\ln 3 + C}$$

Answer: $\boxed{\sqrt{x^2-2x-8} + \ln|x-1+\sqrt{x^2-2x-8}| + C}$

(b) (7 points) $\int \ln(1+x^2)\,dx$

1) Integration by Parts:

$u = \ln(1+x^2)$ $\quad dv = dx$

$du = \frac{2x}{1+x^2}\,dx$ $\quad v = x$

2) $\frac{2x^2}{1+x^2} = \frac{2(1+x^2)-2}{1+x^2} = 2 - \frac{2}{1+x^2}$

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - 2x + 2\int \frac{1}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - 2x + 2\arctan(x) + C$$

Answer: $\boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}(x) + C}$

2. (14 total points) Evaluate the following integrals. Show your work.

(a) (7 points) $\int_1^2 \frac{2}{x^3+x}\,dx$

① Partial Fractions: $\frac{2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$2 = A(x^2+1) + (Bx+C)x$

$2 = (A+B)x^2 + Cx + A$

$\therefore \begin{cases} A=2 \\ C=0 \\ A+B=0 \end{cases} \Rightarrow \begin{cases} A=2 \\ C=0 \\ B=-A=-2 \end{cases}$

$\frac{2}{x(x^2+1)} = \frac{2}{x} - \frac{2x+0}{x^2+1}$ ✓

$= \int_1^2 \left(\frac{2}{x} - \frac{2x}{x^2+1}\right) dx$

$= 2\ln|x|\,\Big|_1^2 - \int_1^2 \frac{2x}{x^2+1}\,dx$

$= 2\ln 2 - 2\cancel{\ln 1}_{0} - \int_2^5 \frac{1}{u}\,du$

② u-sub: $u = x^2+1$, $du = 2x\,dx$

$\left(\begin{matrix} x=1 \Rightarrow u = 1^2+1 = 2 \\ x=2 \Rightarrow u = 2^2+1 = 5 \end{matrix}\right)$

$= 2\ln 2 - \ln|u|\,\Big|_2^5$

$= 2\ln 2 - \ln 5 + \ln 2$

$= 3\ln 2 - \ln 5$

Answer: $\boxed{3\ln 2 - \ln 5 = \ln(8/5)}$

(b) (7 points) $\int_0^2 xe^{x^2} + x^{\sqrt{2}}\,dx$

First integral only: u-sub

$u = x^2$

$\frac{1}{2}du = \frac{\cancel{2}}{\cancel{2}}x\,dx$

$= \int_0^2 x e^{x^2}\,dx + \int_0^2 x^{\sqrt{2}}\,dx$

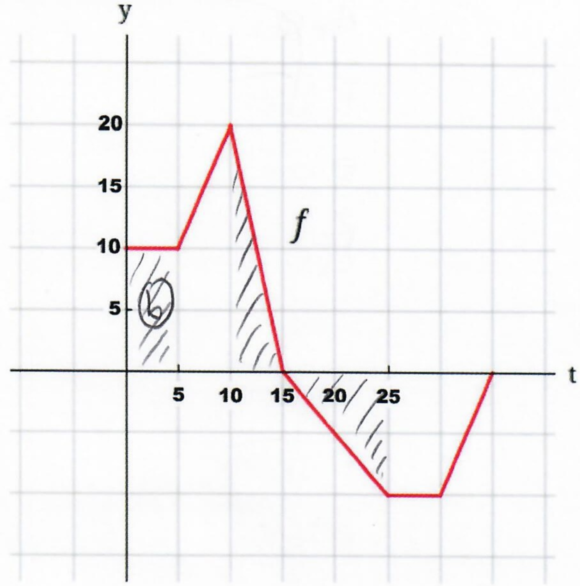
$= \int_{u=0}^{u=4} e^u \frac{1}{2}\,du + \frac{x^{\sqrt{2}+1}}{\sqrt{2}+1}\Big|_0^2$

$= \frac{1}{2}e^u\Big|_0^4 + \frac{2^{\sqrt{2}+1}}{\sqrt{2}+1}$

$= \frac{1}{2}e^4 - \frac{1}{2} + \frac{1}{\sqrt{2}+1}2^{\sqrt{2}+1}$

Answer: $\boxed{\frac{1}{2}(e^4-1) + \frac{1}{\sqrt{2}+1}2^{\sqrt{2}+1}}$

3. (10 total points) Consider the graph of the function f given below consisting of several line segments. Notice that the grid lines are in multiples of 5.



Use the graph to answer the questions below.

(a) (3 points) Find $\int_{10}^{25} f(t)\,dt$ = signed area between graph & x-axis on [10, 25]

$= \frac{1}{2}(20)(5) - \frac{1}{2}(10)(10) = 50 - 50$

Answer = 0

(b) (2 points) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\cdot\left(\frac{5}{n}\right)$ $= \int_0^5 f(x)\,dx =$ area $\boxed{5}^{10}$ $= 50$

Answer = 50

(c) (2 points) If $g(x) = \int_0^x f(t)\,dt$, at what value of x is $g(x)$ maximum?

Answer: $x = 15$

(d) (3 points) If $g(x) = \int_0^x f(t)\,dt$, then find $g''(7)$.

$g'(x) = f(x)$

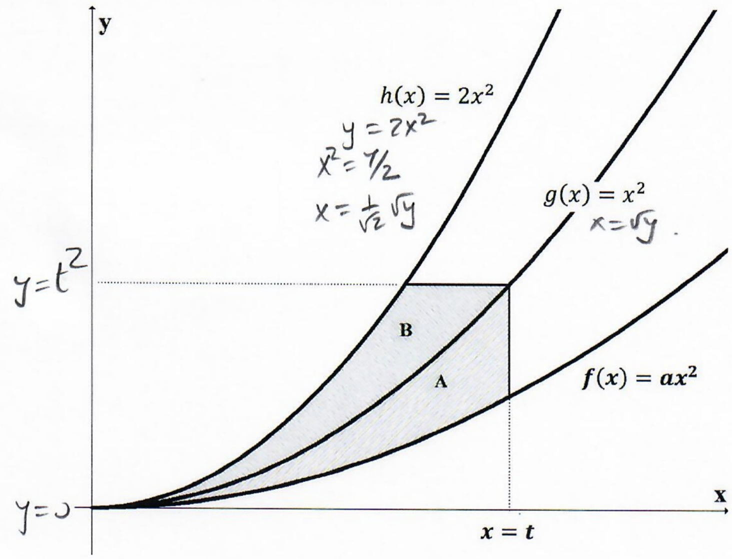
$g''(x) = f'(x) \Rightarrow g''(7) = f'(7) =$ (slope of $f(x)$ at $x=7$) $= \frac{20-10}{10-5} = \frac{10}{5} = 2$

Answer: $g''(7) = 2$

4. (10 points) Let $g(x) = x^2$ and $h(x) = 2x^2$. Assume $f(x) = ax^2$, for some constant $0 < a < 1$.

A vertical line is drawn at some positive number $x = t$ and a horizontal line is drawn through the point where $x = t$ intersects $g(x)$ (as shown). Let A and B be the regions as shown.

If the areas of A and B are the same for all positive values of t, what is the value of a?



$$\text{Area } A = \int_0^t (x^2 - ax^2)\,dx$$
$$= (1-a)\int_0^t x^2\,dx$$
$$= \frac{1}{3}(1-a)x^3\Big|_0^t$$
$$\boxed{\text{Area } A = \frac{1}{3}(1-a)t^3}$$

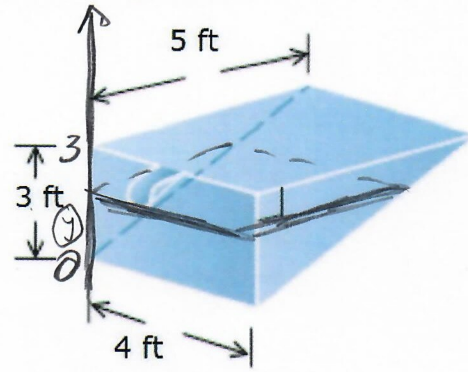
$$\text{Area } B = \int_{y=0}^{y=t^2} \sqrt{y} - \frac{1}{\sqrt{2}}\sqrt{y}\,dy$$
$$= \left(1 - \frac{1}{\sqrt{2}}\right)\int_0^{t^2}\sqrt{y}\,dy = \left(\frac{\sqrt{2}-1}{\cancel{\sqrt{2}}}\right)\frac{\cancel{2}^{\sqrt{2}}}{3}y^{3/2}\Big|_0^{t^2} = \frac{(2-\sqrt{2})}{3}\left((t^2)^{3/2} - 0\right)$$
$$\boxed{\text{Area } B = \frac{2-\sqrt{2}}{3}t^3}$$

We want: Area A = Area B for all $t > 0$

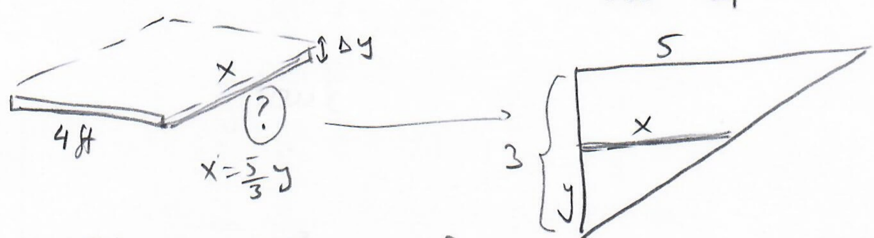
$$\frac{1}{\cancel{3}}(1-a)\cancel{t^3} = \frac{2-\sqrt{2}}{\cancel{3}}\cancel{t^3}$$
$$1 - a = 2 - \sqrt{2}$$
$$1 + \sqrt{2} - 2 = a$$
$$a = \sqrt{2} - 1$$

ANSWER: $a = \boxed{\sqrt{2} - 1}$

5. (10 points) The tank shown below is full of water. Find the work (in ft-lb) required to pump the water out of the spout which is at the top of the tank. Use the fact that water weighs 62.5 lb/ft^3.



With origin at the bottom of tank:
A very thin slice of water at height y above the lowest point of the tank is rectangular with dimensions:



So volume $V_i = 4\left(\frac{5}{3}y\right)\Delta y = \frac{20}{3}y\,\Delta y$ ft^3

$\frac{x}{5} = \frac{y}{3} \Rightarrow x = \frac{5}{3}y$.

and weight/force $F_i \cong 62.5 V_i = 62.5\,\frac{20}{3}y\,\Delta y$

It needs to be lifted $d_i \cong 3-y$

so it needs a work $W_i \cong F_i d_i = 62.5\,\frac{20}{3}y(3-y)\Delta y$.

Adding up & taking limit as nb. of slices $\to \infty$:

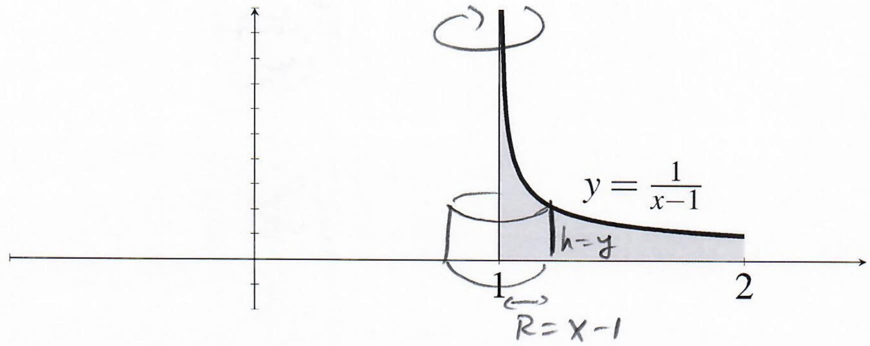
$$\text{Total Work} = \int_0^3 62.5\,\frac{20}{3}y(3-y)\,dy = \frac{1250}{3}\int_0^3 3y-y^2\,dy$$

$$= \frac{1250}{3}\left(\frac{3}{2}y^2-\frac{1}{3}y^3\right)\Big|_0^3$$

$$= \frac{1250}{3}\left(\frac{27}{2}-\frac{27}{3}\right) = \frac{1250}{3}\cdot\frac{27}{6} = 1875$$

ANSWER: Work = 1875 ft-lbs

6. (10 total points) Consider the region $\mathcal{R}$, bounded by the line $x = 1$, the x axis between 1 and 2, and the curve $y = \frac{1}{x-1}$.



(a) (5 points) Does $\mathcal{R}$ have finite area? Justify your answer by computing an improper integral.

$$\text{area}(\mathcal{R}) = \int_1^2 \frac{1}{x-1}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{x-1}\,dx =$$

$$= \lim_{t\to 1^+} \ln|x-1| \Big|_t^2 = \lim_{t\to 1^+} \left(\cancel{\ln 1} - \ln|t-1|\right)$$

$$= -(-\infty) = +\infty$$

ANSWER: Area is ∞ (write "infinite" or computed value)

(b) (5 points) The solid you obtain by rotating the region $\mathcal{R}$ around the line $x = 1$ has finite volume. Compute it.

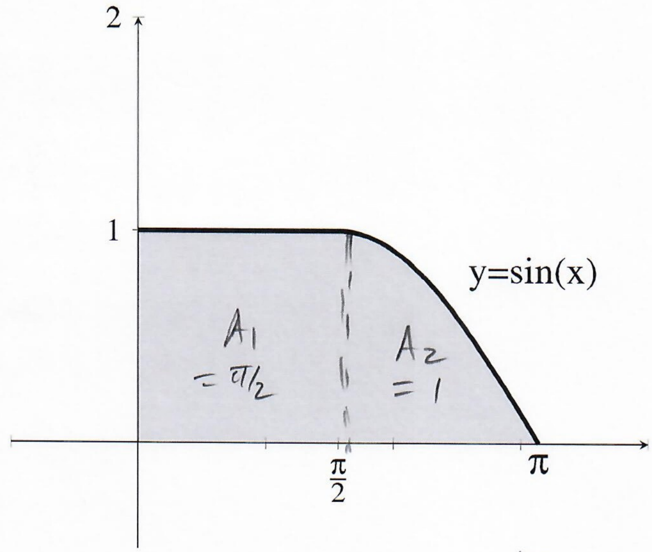
$$\text{Vol}(\mathcal{R}) = \int_1^2 2\pi\,(\text{radius})(\text{height})\,dx = \int_1^2 2\pi\,\cancel{(x-1)}\,\frac{1}{\cancel{x-1}}\,dx.$$

$$= 2\pi\left(x\Big|_1^2\right) = 2\pi$$

ANSWER: Volume = 2π cubic units

7. (10 points) Find the y-coordinate $\bar{y}$ of the centroid of the shaded region below.

The boundary of the region consists of: the y-axis from 0 to 1, the line $y = 1$ from $x = 0$ to $x = \frac{\pi}{2}$, the curve $y = \sin x$ from $x = \frac{\pi}{2}$ to $x = \pi$, and the x axis from 0 to π.



$$\text{Area } A = A_1 + A_2$$
$$= \frac{\pi}{2} + \int_{\pi/2}^{\pi} \sin x \, dx$$
$$= \frac{\pi}{2} - \cos x \Big|_{\pi/2}^{\pi}$$
$$= \frac{\pi}{2} + \left(-\cos\pi + \cos\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2} + 1$$

$$\bar{y} = \frac{A_1 \cdot \bar{y}_1 + A_2 \cdot \bar{y}_2}{A_1 + A_2}$$

$$= \frac{\frac{\pi}{2} \cdot \frac{1}{2} + 1 \cdot \frac{\pi}{8}}{\frac{\pi}{2} + 1}$$

$$= \frac{\pi/4 + \pi/8}{\frac{\pi+2}{2}}$$

$$= \frac{2\pi + \pi}{4\pi + 8} = \boxed{\frac{3\pi}{4\pi+8}}$$

$$\cong 0.458$$

$\bar{y}_1 = \frac{1}{2}$ by symmetry

$$\bar{y}_2 = \frac{1}{A_2} \int_{\pi/2}^{\pi} \frac{1}{2}(\sin^2 x)\,dx$$
$$= \frac{1}{1} \int_{\pi/2}^{\pi} \frac{1}{2} \, \frac{1 - \cos(2x)}{2}\,dx$$
$$= \frac{1}{4}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_{\pi/2}^{\pi}$$
$$= \frac{1}{4} \, \frac{\pi}{2} = \frac{\pi}{8}$$

ANSWER: $\bar{y} = \boxed{\frac{3\pi}{4\pi+8}}$

$\cong$

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = \frac{\cos^3 x}{2y}, \quad y(\pi) = -4$$

For full credit, write your answer in explicit form, $y = f(x)$.

Separating the variables and integrating:

$$\int 2y\,dy = \int \cos^3 x\,dx$$

$$y^2 = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$

$$= \int 1-u^2\,du = u - \frac{1}{3}u^3 + C$$

$$y^2 = \sin x - \frac{1}{3}\sin^3 x + C$$

$$y = \pm\sqrt{\sin x - \frac{1}{3}\sin^3 x + C}$$

$$y(\pi) = -4 \Rightarrow \quad y = - \sqrt{\sin x - \frac{1}{3}\sin^3 x + C}$$

Constant: $y(\pi) = -4 \Rightarrow$

$$-4 = -\sqrt{0 - \frac{1}{3}(0)^3 + C}$$

$$-4 = -\sqrt{C}$$

$$C = 16$$

ANSWER: $y = \boxed{-\sqrt{\sin(x) - \frac{1}{3}\sin^3(x) + 16}}$

9. (12 total points) A tank initially contains 1,000 L of saltwater, with 100 kg of dissolved salt in it. Saltwater containing 0.5 kg/L of dissolved salt comes into the tank at a rate of 4 L/sec. The solution is kept thoroughly mixed and drains from the tank at the same rate of 4 L/sec.

Let $y = y(t)$ be the amount of salt (in kg) in the tank after t seconds.

(a) (3 points) Write the differential equation for $y = y(t)$.

dy/dt = rate in − rate out $= (0.5\text{ kg/L})(4\text{ L/sec}) - \left(\frac{y}{1000}\right)(4\text{ L/sec})$

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{250}} \qquad y(0) = 100\text{ kg}.$$

(b) (7 points) Solve the differential equation and give $y(t)$, including finding all constants of integration.

$$\frac{dy}{dt} = \frac{500 - y}{250} = (y-500)\left(-\frac{1}{250}\right)$$

$$\int \frac{1}{y-500}\,dy = \int -\frac{1}{250}\,dt$$

$$\ln|y-500| = -\frac{t}{250} + C$$

$$|y-500| = e^{C} e^{-t/250} = A e^{-t/250}, \quad (A = e^{C})$$

$$y - 500 = B e^{-t/250} \quad (B = \pm A)$$

$$y = 500 + B e^{-t/250}$$

$y(0) = 100$:

$100 = 500 + Be^{0}$

$B = -400$.

Answer: $y(t) = \boxed{500 - 400e^{-t/250}}$

(c) (2 points) Find $\lim_{t\to\infty} y(t)$.

$$\lim_{t\to\infty} \left(500 - 400\underbrace{e^{-t/250}}_{\to 0}\right) = 500 -$$

Answer: $\lim_{t\to\infty} y(t) = \boxed{500}$ kg