

1. (14 total points) Evaluate the following integrals. Show your work.

(a) (7 points) $\int \frac{x}{\sqrt{x^2-2x-8}} dx = \int \frac{x}{\sqrt{(x-1)^2-9}} dx$

$$= \int \frac{3 \sec \theta + 1}{\sqrt{9 \sec^2 \theta - 9}} 3 \tan \theta \sec \theta d\theta$$

$$= \int \frac{3 \sec \theta + 1}{3 \tan \theta} 3 \tan \theta \sec \theta d\theta$$

$$= \int 3 \sec^2 \theta + \sec \theta d\theta$$

$$= 3 \tan \theta + \ln | \sec \theta + \tan \theta | + C$$

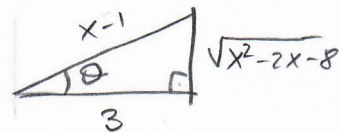
$$= \cancel{3} \frac{\sqrt{x^2-2x-8}}{\cancel{3}} + \ln \left| \frac{x-1}{3} + \frac{\sqrt{x^2-2x-8}}{3} \right| + C$$

$$= \sqrt{x^2-2x-8} + \ln | x-1 + \sqrt{x^2-2x-8} | - \ln 3 + C$$

Trig Sub:

$$x-1 = 3 \sec \theta$$

$$dx = 3 \tan \theta \sec \theta d\theta$$



$$\sec \theta = \frac{x-1}{3} = \frac{\text{hyp}}{\text{adj}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-2x-8}}{3}$$

Answer:

$$\sqrt{x^2-2x-8} + \ln | x-1 + \sqrt{x^2-2x-8} | + C$$

(b) (7 points) $\int \ln(1+x^2) dx$

$$= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx$$

$$= x \ln(1+x^2) - \int 2 - \frac{2}{1+x^2} dx$$

$$= x \ln(1+x^2) - 2x + 2 \int \frac{1}{1+x^2} dx$$

$$= x \ln(1+x^2) - 2x + 2 \arctan(x) + C$$

1) Integration by Parts:

$$u = \ln(1+x^2) \quad dv = dx$$

$$du = \frac{2x}{1+x^2} dx \quad v = x$$

$$2) \frac{2x^2}{1+x^2} = \frac{2(1+x^2)-2}{1+x^2} = 2 - \frac{2}{1+x^2}$$

Answer:

$$x \ln(1+x^2) - 2x + 2 \tan^{-1}(x) + C$$

2. (14 total points) Evaluate the following integrals. Show your work.

(a) (7 points) $\int_1^2 \frac{2}{x^3+x} dx$

$$= \int_1^2 \left(\frac{2}{x} - \frac{2x}{x^2+1} \right) dx$$

$$= 2 \ln|x| \Big|_1^2 - \int_1^2 \frac{2x}{x^2+1} dx$$

$$= 2 \ln 2 - 2 \ln 1 - \int_2^5 \frac{1}{u} du$$

$$= 2 \ln 2 - \ln|u| \Big|_2^5$$

$$= 2 \ln 2 - \ln 5 + \ln 2$$

$$= 3 \ln 2 - \ln 5$$

① Partial Fractions: $\frac{2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$$2 = A(x^2+1) + (Bx+C)x$$

$$2 = (A+B)x^2 + Cx + A$$

$$\therefore \begin{cases} A=2 \\ C=0 \\ A+B=0 \end{cases} \Rightarrow \begin{cases} A=2 \\ C=0 \\ B=-A=-2 \end{cases}$$

$$\frac{2}{x(x^2+1)} = \frac{2}{x} - \frac{2x+0}{x^2+1} \quad \checkmark$$

② u-sub: $u = x^2+1$
 $du = 2x dx$

$$\left(\begin{array}{l} x=1 \Rightarrow u=1^2+1=2 \\ x=2 \Rightarrow u=2^2+1=5 \end{array} \right)$$

Answer:

$$3 \ln 2 - \ln 5 = \ln\left(\frac{8}{5}\right)$$

(b) (7 points) $\int_0^2 x e^{x^2} + x^{\sqrt{2}} dx$

$$= \int_0^2 x e^{x^2} dx + \int_0^2 x^{\sqrt{2}} dx$$

$$\equiv \int_{u=0}^{u=4} e^u \frac{1}{2} du + \frac{x^{\sqrt{2}+1}}{\sqrt{2}+1} \Big|_0^2$$

$$= \frac{1}{2} e^u \Big|_0^4 + \frac{2^{\sqrt{2}+1}}{\sqrt{2}+1}$$

$$= \frac{1}{2} e^4 - \frac{1}{2} + \frac{1}{\sqrt{2}+1} 2^{\sqrt{2}+1}$$

First integral only: u-sub

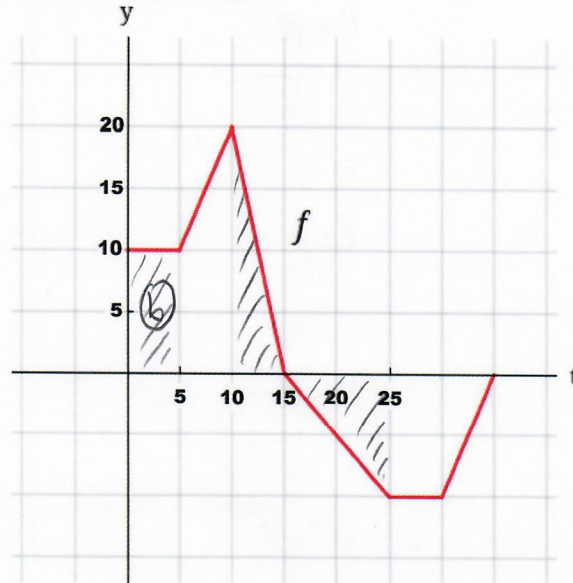
$$u = x^2$$

$$\frac{1}{2} du = x dx$$

Answer:

$$\frac{1}{2} (e^4 - 1) + \frac{1}{\sqrt{2}+1} 2^{\sqrt{2}+1}$$

3. (10 total points) Consider the graph of the function f given below consisting of several line segments. Notice that the grid lines are in multiples of 5.



Use the graph to answer the questions below.

- (a) (3 points) Find $\int_{10}^{25} f(t) dt = \text{signed area between graph \& x-axis on } [10, 25]$
 $= \frac{1}{2} (20)(5) - \frac{1}{2} (10)(10) = 50 - 50$

Answer =

0

- (b) (2 points) Evaluate $\lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{5i}{n}\right) \cdot \left(\frac{5}{n}\right) = \int_0^5 f(x) dx = \text{area } [5]^{10} = 50$

Answer =

50

- (c) (2 points) If $g(x) = \int_0^x f(t) dt$, at what value of x is $g(x)$ maximum?

Answer: $x =$

15

- (d) (3 points) If $g(x) = \int_0^x f(t) dt$, then find $g''(7)$.

$$g'(x) = f(x) \quad \rightarrow \quad g''(7) = f'(7) = (\text{slope of } f(x) \text{ at } x=7) = \frac{20-10}{10-5} = \frac{10}{5} = 2$$

$$g''(x) = f'(x) \rightarrow$$

Answer: $g''(7) =$

2

4. (10 points) Let $g(x) = x^2$ and $h(x) = 2x^2$. Assume $f(x) = ax^2$, for some constant $0 < a < 1$.

A vertical line is drawn at some positive number $x = t$ and a horizontal line is drawn through the point where $x = t$ intersects $g(x)$ (as shown). Let A and B be the regions as shown.

If the areas of A and B are the same for all positive values of t , what is the value of a ?

$$\begin{aligned} \text{Area } A &= \int_0^t (x^2 - ax^2) dx \\ &= (1-a) \int_0^t x^2 dx \\ &= \frac{1}{3}(1-a)x^3 \Big|_0^t \end{aligned}$$

$$\boxed{\text{Area } A = \frac{1}{3}(1-a)t^3}$$

$$\text{Area } B = \int_{y=0}^{y=t^2} \sqrt{y} - \frac{1}{\sqrt{2}}\sqrt{y} dy$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) \int_0^{t^2} \sqrt{y} dy = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{\sqrt{2}}{3} y^{3/2} \Big|_0^{t^2} = \frac{(2-\sqrt{2})}{3} \left((t^2)^{3/2} - 0 \right)$$

$$\boxed{\text{Area } B = \frac{2-\sqrt{2}}{3} t^3}$$

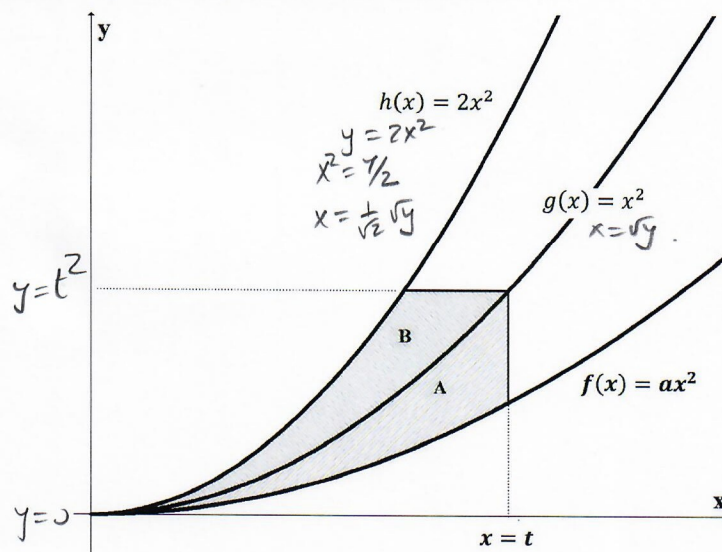
We want: Area $A =$ Area B for all $t > 0$

$$\frac{1}{3}(1-a)t^3 = \frac{2-\sqrt{2}}{3} t^3$$

$$1-a = 2-\sqrt{2}$$

$$1+\sqrt{2}-2 = a$$

$$\boxed{a = \sqrt{2}-1}$$



ANSWER: $a =$

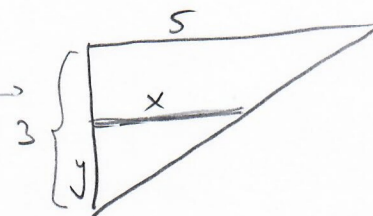
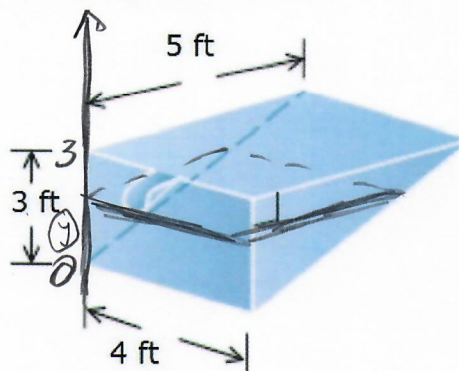
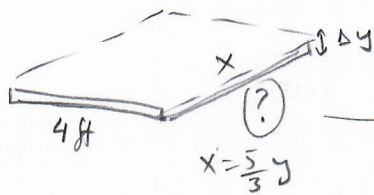
$$\boxed{\sqrt{2}-1}$$

5. (10 points) The tank shown below is full of water. Find the work (in ft-lb) required to pump the water out of the spout which is at the top of the tank. Use the fact that water weighs 62.5 lb/ft^3 .

With origin at the bottom of tank:

A very thin slice of water at height y above the lowest point of the tank is rectangular with

dimensions:



$$\text{So volume } V_i = 4 \left(\frac{5}{3} y \right) \Delta y = \frac{20}{3} y \Delta y \text{ ft}^3$$

$$\text{and weight/force } F_i \approx 62.5 V_i = 62.5 \frac{20}{3} y \Delta y$$

$$\frac{x}{5} = \frac{y}{3} \Rightarrow x = \frac{5}{3} y$$

It needs to be lifted $d_i \approx 3 - y$

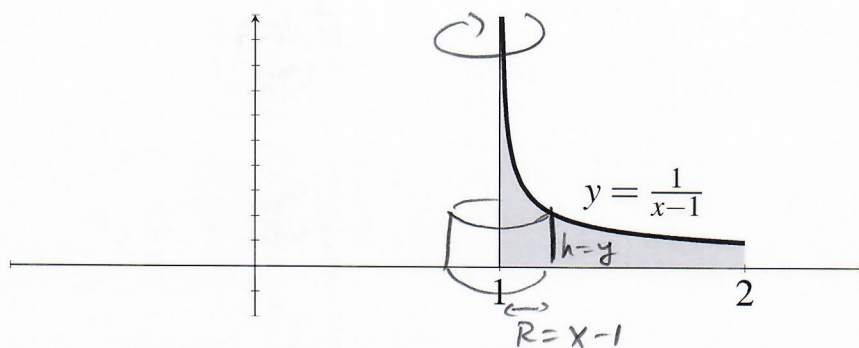
$$\text{so it needs a work } W_i \approx F_i d_i = 62.5 \frac{20}{3} y (3 - y) \Delta y$$

Adding up & taking limit as nb. of slices $\rightarrow \infty$:

$$\begin{aligned} \text{Total Work} &= \int_0^3 62.5 \frac{20}{3} y (3 - y) dy = \frac{1250}{3} \int_0^3 (3y - y^2) dy \\ &= \frac{1250}{3} \left(\frac{3}{2} y^2 - \frac{1}{3} y^3 \right) \Big|_0^3 \\ &= \frac{1250}{3} \left(\frac{27}{2} - \frac{27}{3} \right) = \frac{1250}{3} \cdot \frac{27}{6} = 1875 \end{aligned}$$

ANSWER: Work = 1875 ft-lbs

6. (10 total points) Consider the region \mathcal{R} , bounded by the line $x = 1$, the x axis between 1 and 2, and the curve $y = \frac{1}{x-1}$.



- (a) (5 points) Does \mathcal{R} have finite area? Justify your answer by computing an improper integral.

$$\begin{aligned} \text{area}(\mathcal{R}) &= \int_1^2 \frac{1}{x-1} dx = \lim_{t \rightarrow 1^+} \int_t^2 \frac{1}{x-1} dx = \\ &= \lim_{t \rightarrow 1^+} \ln|x-1| \Big|_t^2 = \lim_{t \rightarrow 1^+} (\ln 1 - \ln|t-1|) \\ &= -(-\infty) = +\infty \end{aligned}$$

ANSWER: Area is ∞ (write "infinite" or computed value)

- (b) (5 points) The solid you obtain by rotating the region \mathcal{R} around the line $x = 1$ has finite volume. Compute it.

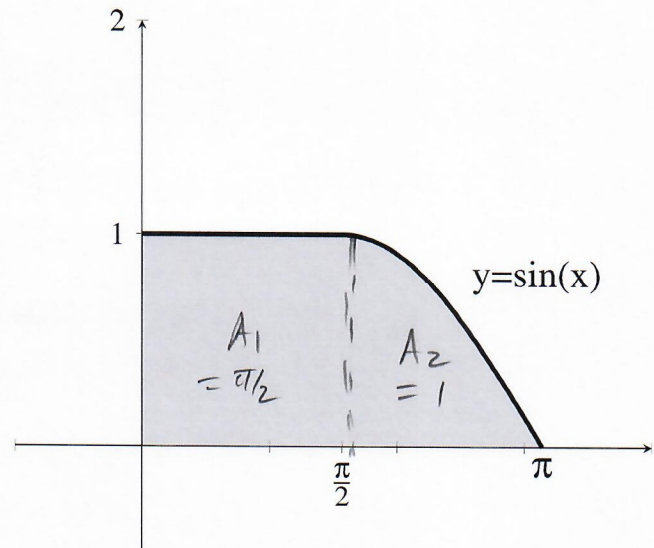
$$\begin{aligned} \text{Vol}(\mathcal{R}) &= \int_1^2 2\pi (\text{radius})(\text{height}) dx = \int_1^2 2\pi (x-1) \frac{1}{x-1} dx \\ &= 2\pi (x \Big|_1^2) = 2\pi \end{aligned}$$

ANSWER: Volume = 2π cubic units

7. (10 points) Find the y-coordinate \bar{y} of the centroid of the shaded region below.

The boundary of the region consists of: the y-axis from 0 to 1, the line $y = 1$ from $x = 0$ to $x = \frac{\pi}{2}$, the curve $y = \sin x$ from $x = \frac{\pi}{2}$ to $x = \pi$, and the x axis from 0 to π .

$$\begin{aligned} \text{Area } A &= A_1 + A_2 \\ &= \frac{\pi}{2} + \int_{\pi/2}^{\pi} \sin x \, dx \\ &= \frac{\pi}{2} - \cos x \Big|_{\pi/2}^{\pi} \\ &= \frac{\pi}{2} + (-\cos \pi + \cos \frac{\pi}{2}) \\ &= \frac{\pi}{2} + 1 \end{aligned}$$



$$\bar{y} = \frac{A_1 \cdot \bar{y}_1 + A_2 \cdot \bar{y}_2}{A_1 + A_2}$$

$$= \frac{\frac{\pi}{2} \cdot \frac{1}{2} + 1 \cdot \frac{\pi}{8}}{\frac{\pi}{2} + 1}$$

$$= \frac{\pi/4 + \pi/8}{\frac{\pi+2}{2}}$$

$$= \frac{2\pi + \pi}{4\pi + 8} = \boxed{\frac{3\pi}{4\pi + 8}}$$

$$\approx 0.458$$

$$\bar{y}_1 = \frac{1}{2} \text{ by symmetry}$$

$$\begin{aligned} \bar{y}_2 &= \frac{1}{A_2} \int_{\pi/2}^{\pi} \frac{1}{2} (\sin^2 x) \, dx \\ &= \frac{1}{1} \int_{\pi/2}^{\pi} \frac{1}{2} \frac{1 - \cos(2x)}{2} \, dx \\ &= \frac{1}{4} \left(x - \frac{1}{2} \sin(2x) \right) \Big|_{\pi/2}^{\pi} \\ &= \frac{1}{4} \frac{\pi}{2} = \frac{\pi}{8} \end{aligned}$$

$$\text{ANSWER: } \bar{y} = \boxed{\frac{3\pi}{4\pi + 8}}$$

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dx} = \frac{\cos^3 x}{2y}, \quad y(\pi) = -4$$

For full credit, write your answer in explicit form, $y = f(x)$.

Separating the variables and integrating:

$$\int 2y \, dy = \int \cos^3 x \, dx$$

$$y^2 = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int 1 - u^2 \, du = u - \frac{1}{3} u^3 + C$$

$$y^2 = \sin x - \frac{1}{3} \sin^3 x + C$$

$$y = \pm \sqrt{\sin x - \frac{1}{3} \sin^3 x + C}$$

$$y(\pi) = -4 \Rightarrow y = \ominus \sqrt{\sin x - \frac{1}{3} \sin^3 x + C}$$

$$\begin{aligned} \text{Constant: } y(\pi) = -4 \Rightarrow -4 &= -\sqrt{0 - \frac{1}{3}(0)^3 + C} \\ -4 &= -\sqrt{C} \\ C &= 16 \end{aligned}$$

ANSWER: $y = \boxed{-\sqrt{\sin(x) - \frac{1}{3} \sin^3(x) + 16}}$

9. (12 total points) A tank initially contains 1,000 L of saltwater, with 100 kg of dissolved salt in it. Saltwater containing 0.5 kg/L of dissolved salt comes into the tank at a rate of 4 L/sec. The solution is kept thoroughly mixed and drains from the tank at the same rate of 4 L/sec.

Let $y = y(t)$ be the amount of salt (in kg) in the tank after t seconds.

- (a) (3 points) Write the differential equation for $y = y(t)$.

$$dy/dt = \text{rate in} - \text{rate out} = (0.5 \text{ kg/L})(4 \text{ L/sec}) - \left(\frac{y}{1000}\right)(4 \text{ L/sec})$$

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{250}}$$

$$y(0) = 100 \text{ kg.}$$

- (b) (7 points) Solve the differential equation and give $y(t)$, including finding all constants of integration.

$$\frac{dy}{dt} = \frac{500 - y}{250} = (y - 500) \left(-\frac{1}{250}\right)$$

$$\int \frac{1}{y-500} dy = \int -\frac{1}{250} dt$$

$$\ln |y-500| = -\frac{t}{250} + C$$

$$|y-500| = e^C e^{-t/250} = A e^{-t/250}, \quad (A = e^C)$$

$$y-500 = B e^{-t/250}, \quad (B = \pm A)$$

$$y = 500 + B e^{-t/250}$$

$$y(0) = 100:$$

$$100 = 500 + B e^0$$

$$B = -400$$

$$\text{Answer: } \boxed{y(t) = 500 - 400 e^{-t/250}}$$

- (c) (2 points) Find $\lim_{t \rightarrow \infty} y(t)$.

$$\lim_{t \rightarrow \infty} (500 - 400 e^{-t/250}) = 500 - 400 = 100$$

$$\text{Answer: } \boxed{\lim_{t \rightarrow \infty} y(t) = 100 \text{ kg}}$$