1. (12 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) \[ \int_0^{\pi/4} y \sin(y) \, dy \]

Integration by Parts: \( u = y \) \( \quad \& \quad dv = \sin(y) \, dy \)
\( du = dy \) \( \quad \& \quad v = -\cos(y) \)

\[ = -y \cos(y) \bigg|_0^{\pi/4} + \int_0^{\pi/4} \cos(y) \, dy \]

\[ = \left[ -y \cos(y) + \sin(y) \right]_0^{\pi/4} \]

\[ = \left[ -\pi \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right] - \left[ 0 \right] \]

\[ = -\pi \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \left( 1 - \pi \right) \]

(b) \[ \int_0^{\pi/6} 16 \cos^2(2x) \sin^2(2x) \, dx \]

\[ = \int_0^{\pi/6} 16 \left( \frac{1 + \cos(4x)}{2} \right) \left( \frac{1 - \cos(4x)}{2} \right) \, dx \]

\[ = \int_0^{\pi/6} 4 \left( 1 - \cos^2(4x) \right) \, dx \]

\[ = \int_0^{\pi/6} 4 \left( \frac{1 - \cos(8x)}{2} \right) \, dx \]

\[ = \left[ 2x - \frac{1}{4} \sin(8x) \right]_0^{\pi/6} \]

\[ = \left[ \frac{\pi}{3} - \frac{1}{4} \sin\left( \frac{4\pi}{3} \right) \right] - \left[ 0 \right] \]

\[ = \frac{\pi}{3} - \frac{1}{4} \cdot \frac{-\sqrt{3}}{2} \]

\[ = \frac{\pi}{3} + \frac{\sqrt{3}}{8} \]
2. (14 points) Answer the following two unrelated questions. Show your work and box your answer.

(a) Evaluate the integral: \( \int \ln(x^2 + 1) \, dx \)

\[
\begin{align*}
\int \ln(x^2 + 1) \, dx &= \int \frac{2x}{x^2 + 1} \, dx \\
&= x \ln(x^2 + 1) - \int \frac{2x}{x^2 + 1} \, dx \\
&= x \ln(x^2 + 1) - \ln(x^2 + 1) - 2\arctan(x) + C
\end{align*}
\]

(b) The acceleration and the initial velocity of a object moving on a straight line are given by:

\[ a(t) = 2t + 6 \text{ m/s}^2 \quad \text{and} \quad v(0) = -7 \text{ m/s} \]

Find the total distance traveled by the particle from \( t = 0 \) to \( t = 2 \) seconds.

\[
\begin{align*}
\text{Accel.} &= 2t + 6 \quad \Rightarrow \quad \text{Vel.} = t^2 + 6t + C \\
\text{Since} \quad v(0) = -7 \quad \Rightarrow \quad C = -7 \\
\text{So:} \quad \langle v(t) = t^2 + 6t - 7 \text{ m/s} \rangle &= \langle (t+7)(t-1) \rangle \\
\text{Total dist.} &= \int_0^2 \left| t^2 + 6t - 7 \right| \, dt \\
&= \int_0^1 \left( -t^2 - 6t + 7 \right) \, dt + \int_1^2 \left( t^2 + 6t - 7 \right) \, dt \\
&= \left[ -\frac{1}{3}t^3 - 3t^2 + 7t \right]_0^1 + \left[ \frac{1}{3}t^3 + 3t^2 - 7t \right]_1^2 \\
&= \left[ -\frac{11}{3} \right] + \left( \frac{2}{3} - \frac{-11}{3} \right) \\
&= \frac{8}{3} \text{ meters}
\end{align*}
\]
3. Consider the region enclosed by the graphs \( y = 9x - x^3 \), \( y = 2x \), \( x = 0 \), and \( x = 2 \) pictured below.

(a) (4 points) **Set up** an integral that represents the volume of the solid formed by rotating this region about the y-axis. (Do not compute the volume).

Note: We cannot solve \( y = 9x - x^3 \) for \( x \) in terms of \( y \) so we must set up our integrals in \( x \). This means vertical rectangles, so shells for part (a) & washers in part (b).

\[
V_1 = \int_0^2 2\pi x \left[ (9x - x^3) - (2x) \right] \, dx
\]

\[
= \int_0^2 2\pi x (7x - x^3) \, dx
\]

(b) (4 points) **Set up** an integral that represents the volume of the solid formed by rotating this region about the line \( y = -5 \). (Do not compute the volume).

Washers:

\[
V_2 = \int_0^2 \pi R^2 - \pi r^2 \, dx
\]

\[
= \left[ \int_0^2 \pi \left[ (9x - x^3)^2 - (2x + 5)^2 \right] \, dx \right]
\]
4. (7 points) The graph below shows the instantaneous velocity \( v(t) \) (in meters per second) of an object moving along a straight line, as a function of time \( t \) (in seconds).

Use Simpson's Rule with \( n = 6 \) subintervals to approximate the **average velocity** \( v_{\text{ave}} \) of the object from \( t = 0 \) to \( t = 6 \) seconds.

\[
v_{\text{ave}} = \frac{1}{6} \int_{0}^{6} v(t) \, dt
\]

\[
= \frac{1}{6} \cdot \frac{1}{3} \left[ v(0) + 4v(1) + 2v(2) + 4v(3) + 2v(4) + 4v(5) + v(6) \right]
\]

\[
= \frac{1}{6} \cdot \frac{1}{3} \cdot \left[ 4 + 4(9) + 2(20) + 4(31) + 2(36) + 4(29) + 1 \right]
\]

\[
= \frac{1}{18} \left[ 4 + 36 + 40 + 124 + 72 + 116 + 1 \right] = \frac{1}{18} [396]
\]

\[
= \boxed{22} \text{ m/s}
\]
5. Let \( R \) be the region in the first quadrant which is shown below, and it is described by:

\[
0 \leq y \leq \frac{x}{\sqrt{4-x^2}}, \quad 0 \leq x < 2
\]

Note that \( f(x) = \frac{x}{\sqrt{4-x^2}} \) has a vertical asymptote; use limits for improper integrals as needed, and determine if they converge or diverge.

(a) (6 points) Compute the area of this region \( R \).

\[
A = \int_{0}^{2} \frac{x}{\sqrt{4-x^2}} \, dx
\]

\[
= \int_{0}^{1} \frac{1}{\sqrt{u}} \left( \frac{1}{2} \right) du
\]

\[
= \frac{1}{2} \int_{0}^{1} \frac{du}{\sqrt{u}} = \frac{1}{2} \lim_{a \to 0^+} \int_{a}^{1} u^{-\frac{1}{2}} du.
\]

\[
= \frac{1}{2} \lim_{a \to 0^+} 2\sqrt{u} \bigg|_{a}^{1} = \lim_{a \to 0^+} (\sqrt{1} - \sqrt{a})
\]

\[
= 2
\]

(b) (7 points) Compute the x-coordinate, \( \bar{x} \), of its centroid (center of mass).

\[
\bar{x} = \frac{1}{2} \int_{0}^{2} \frac{x^2}{\sqrt{4-x^2}} \, dx
\]

\[
= \frac{1}{2} \int_{0}^{\pi/2} \frac{4 \sin^2 \theta}{2 \cos \theta} \cdot 2 \cos \theta \, d\theta
\]

\[
= \int_{0}^{\pi/2} \frac{1 - \cos 2\theta}{2} \, d\theta
\]

\[
= \left[ \theta - \frac{1}{2} \sin(2\theta) \right]_{0}^{\pi/2}
\]

\[
= \left[ \frac{\pi}{2} - \frac{1}{2} \sin(\pi) \right] - \left[ 0 \right]
\]

\[
= \frac{\pi}{2}
\]

\[\text{(problem continues on the next page)}\]
(c) (8 points) Recall the region $\mathcal{R}$ from the previous page, bounded above by $y = \frac{x}{\sqrt{4-x^2}}$, for $0 \leq x < 2$.

Use limits for improper integrals as needed, and determine if they converge or diverge.

Compute the $y$-coordinate, $\bar{y}$, of the centroid of $\mathcal{R}$.

\[
\bar{y} = \frac{1}{2} \int_0^2 \frac{1}{2} \left( \frac{x}{\sqrt{4-x^2}} \right)^2 \, dx
\]

\[
= \frac{1}{4} \int_0^2 \frac{x^2}{4-x^2} \, dx.
\]

\[
= \frac{1}{4} \lim_{b \to 2^-} \left[ \int_0^b \frac{x^2}{4-x^2} \, dx \right].
\]

\[
= \frac{1}{4} \lim_{b \to 2^-} \left[ \left. \frac{x^2}{2} \right|_0^b + \ln \left| \frac{2+x}{2-x} \right| \right].
\]

\[
= \frac{1}{4} \left[ -2 + \lim_{b \to 2^-} \ln \left| \frac{2+b}{2-b} \right| \right].
\]

DIVERGES

\[
\int \frac{-x^2}{4-x^2} \, dx
\]

\[
= \int -1 + \frac{1}{4-x^2} \, dx
\]

\[
= -x + \int \frac{1}{(2-x)(2+x)} \, dx
\]

\[
\int P.F. \cdot \frac{1}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}
\]

\[
A = 4 \cdot A(2+x) \Rightarrow A = 1
\]

\[
B = 4 \cdot B(2-x) \Rightarrow B = 1.
\]

\[
= -x + \int \frac{1}{2-x} + \frac{1}{2+x} \, dx
\]

\[
= -x - \ln |2-x| + \ln |2+x| + C
\]

\[
= -x + \ln \left| \frac{2+x}{2-x} \right| + C
\]

Remark: $\int \frac{-x^2}{4-x^2} \, dx$ could also be computed using a trig sub with

$x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta \, d\theta$ 

(Answer: $-x + 2 \ln \left| \frac{2+x}{2-x} \right|$)

However, $x = 2 \sec \theta$ is not correct because the bounds are $0 \leq x \leq 2$

so $\sec \theta = \frac{x}{2}$ would $\leq 1$ which is not possible. You'd also get an impossible right triangle.
6. (8 points) A tank of the shape shown in the picture, with height=7m, length=10m, and width=5m, is full of water. Water weighs 1000 kg/m³, and the gravitational acceleration is \( g = 9.8 \text{ m/s}^2 \).

Set up (do not evaluate) an integral equal to the work required to pump all the water out of the tank through a spout that is 1 m above the top of the tank.

**Specify the meaning of your variable of integration, either in words or on the picture.**

With \( y \) = height above bottom:

\[
W = \int_0^7 9800 (8-y) (5x \, dy)
\]

\[
= \int_0^7 9800 (8-y) \frac{50}{7} (7-y) \, dy
\]

\[(02)\]

With \( y \) = depth below top of tank:

\[
W = \int_0^7 9800 (y+1) \frac{50}{7} y \, dy
\]

7. (a) (4 points) Write down an integral equal to the **arclength** \( L(t) \) of the portion of the curve:

\[ y = e^x, \text{ from } x = 0 \text{ to } x = t. \]

\[
dy/dx = e^{x^2}, 2x
\]

\[
L(t) = \int_0^t \sqrt{1 + (e^{x^2} \cdot 2x)^2} \, dx = \int_0^t \sqrt{1 + 4x^2 e^{2x^2}} \, dx
\]

(b) (4 points) At what rate is \( L(t) \) increasing when \( t = 1 \)?

\[
L'(t) = \sqrt{1 + 4 \cdot t^2 e^{t^2}}
\]

\[
L'(1) = \sqrt{1 + 4 e^2}
\]
8. (10 points) Find the solution to the differential equation

\[
\frac{dy}{dx} = \frac{xy + y}{2 \ln(y)}
\]

that satisfies the initial condition \( y(1) = e^2 \). Give your solution in explicit form, \( y = f(x) \).

\[
\frac{d\ln y}{dx} = \frac{x+1}{2} \frac{\ln y}{x+y} \]

\[
\int \frac{\ln y}{y} \, dy = \int \frac{x+1}{2} \, dx
\]

\[
\frac{\ln y}{y} \, dy = \frac{1}{2} \left( \frac{x^2}{2} + x \right) + C
\]

\[
\frac{u^2}{2} = \frac{1}{2} \left( \frac{x^2}{2} + x \right) + C
\]

\[
(\ln y)^2 = \frac{x^2}{2} + x + C_1
\]

\[
y(1) = e^2 \Rightarrow \frac{(\ln e^2)^2}{4} = \frac{1}{2} + 1 + C_1 \Rightarrow C_1 = 4 - \frac{1}{2} - 1 = 3 - \frac{1}{2}
\]

\[
C_1 = \frac{5}{2}
\]

\[
(\ln y)^2 = \frac{x^2}{2} + x + \frac{5}{2}
\]

\[
\ln y = \frac{1}{2} \sqrt{\frac{x^2}{2} + x + \frac{5}{2}}
\]

\[
y(1) = e^2 \Rightarrow \text{we need the } 0
\]

\[
y = e^{\frac{1}{2} \sqrt{\frac{x^2}{2} + x + \frac{5}{2}}}
\]
9. A 2000 L tank is full of a mixture of water and salt, with 500 grams of salt initial dissolved in the tank. Fresh water (with NO salt) is pumped into the tank at a rate of 20 L/s. The mixture is kept stirred and is pumped out at a rate of 40 L/s. (This means the tank is losing volume at a rate of 20 - 40 = -20 L/s).

(a) (1 point) Give the linear function \( V(t) = at + b \) for the volume in liters after \( t \) seconds.

\[
V(t) = -20t + 2000
\]

(b) (4 points) Let \( y(t) \) be the amount of salt in grams in the tank after \( t \) seconds. Write down the differential equation AND initial condition satisfied by \( y(t) \). Do not solve anything yet.

\[
\frac{dy}{dt} = 0 - \left( \frac{y}{-20t+2000} \right) (40)
\]

\[
y(0) = 500
\]

(c) (6 points) Solve the differential equation to find \( y(t) \). Show work. Simplify and box your answer.

\[
\frac{dy}{dt} = \frac{2y}{t-100} \quad \Rightarrow \quad \int \frac{1}{y} \, dy = \int \frac{2}{t-100} \, dt
\]

\[
\ln |y| = 2 \ln |t-100| + C
\]

\[
y = e^{2 \ln |t-100| + C} = C_1 e^{2 \ln |t-100|} = C_1 (t-100)^2
\]

\[
y(0) = 500: \quad 500 = C_1 |1-100| = C_1 (10000) \quad \Rightarrow \quad C_1 = \frac{500}{10000} = \frac{1}{20}
\]

\[
y = \frac{1}{20} |t-100|^2
\]

(d) (1 point) How many grams of salt are left in the tank after 60 seconds? Simplify your answer.

\[
y(60) = \frac{1}{20} |60-100|^2 = \frac{1}{20} |-40|^2 = \frac{1600}{20} = 80 \text{ grams}
\]