1. (14 points)

(a) (7 points) Evaluate the integral \( \int \frac{1}{x^3 - 4x^2} \, dx \). Show your work, and box your answer.

\[
\int \frac{1}{x^3 - 4x^2} \, dx = \int \frac{1}{x^2(x-4)} \, dx = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} \quad \text{(Partial Fractions)}
\]

Solving for \( A, B, C \):

\[
A = -\frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = \frac{1}{4}, \quad A = -\frac{1}{4}
\]

\[
\therefore \quad \int \frac{1}{x^3 - 4x^2} \, dx = \int \left( -\frac{1}{4x} - \frac{1}{4x^2} + \frac{1}{4} \ln |x-4| + C \right) \, dx
\]

\[
= -\frac{1}{4} \ln |x| - \frac{1}{4} \left( \ln x \right) + \frac{1}{4} \ln |x-4| + C
\]

\[
= -\frac{1}{4} + \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C
\]

(b) (7 points) Evaluate the following improper integral, if it converges, or show why it diverges.

\[
\int_0^\infty \frac{e^x}{1 + e^{2x}} \, dx
\]

\[
\int_0^\infty \frac{e^x}{1 + e^{2x}} \, dx = \int_0^\infty \frac{1}{1 + u^2} \, du = \lim_{t \to \infty} \int_1^t \frac{1}{1 + u^2} \, du
\]

\[
= \lim_{t \to \infty} \left[ \arctan u \right]_1^t
\]

\[
= \lim_{t \to \infty} \left[ \arctan t - \arctan 1 \right]
\]

\[
= \lim_{t \to \infty} \left( \arctan t \right) - \frac{\pi}{4}
\]

\[
= \frac{\pi}{2} - \frac{\pi}{4}
\]

\[
\text{Integral converges to } \frac{\pi}{4}
\]
2. (14 points)

(a) (7 points) Evaluate \( \int_0^{\sqrt{3}} x \tan^{-1}(x) \, dx \).

Give your answer in exact form (in terms of square roots and/or multiples of \( \pi \)).

\[
\begin{align*}
\text{Applying Integration by Parts with } u &= \tan^{-1} x, \quad dv = x \, dx \\
& \Rightarrow du = \frac{1}{1+x^2} \, dx, \quad v = \frac{x^2}{2} \\
\text{we get: } \\
\int_0^{\sqrt{3}} x \tan^{-1} x \, dx &= \frac{x^2}{2} \tan^{-1} x \bigg|_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2} \, dx \\
&= \left( \frac{3}{2} \tan^{-1} \sqrt{3} \bigg|_0 \right) - \frac{1}{2} \int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2} \, dx \\
&= \frac{3\pi}{6} - \frac{1}{2} \left( \sqrt{3} - \tan^{-1} 0 \right) \\
&= \frac{3\pi}{6} - \frac{1}{2} \sqrt{3} + \frac{1}{2} \frac{\pi}{2} \\
&= \frac{2\pi - \sqrt{3}}{2} \\
\end{align*}
\]

(b) (7 points) Find the function \( f(x) \) if \( f'(x) = \frac{1}{(r^2 - x^2)^{3/2}} \) and \( f(0) = 0 \).

The constant \( r \) should appear in your answer.

\[
\begin{align*}
\text{Applying Trig Sub with } x &= r \sin \theta, \quad dx = r \cos \theta \, d\theta, \text{ we get:} \\
\int \frac{1}{(r^2 - x^2)^{3/2}} \, dx &= \int \frac{1}{r^3 \cos^3 \theta} \, r \cos \theta \, d\theta \\
&= \frac{1}{r^2} \int \sec^2 \theta \, d\theta \\
&= \frac{1}{r^2} \tan \theta + C \\
&= \frac{1}{r^2} \frac{x}{\sqrt{r^2 - x^2}} + C \\
\end{align*}
\]

\[
\begin{align*}
f(x) &= \frac{1}{r^2} \frac{x}{\sqrt{r^2 - x^2}} + C \quad \text{and } f(0) = 0 \\
\Rightarrow C &= 0.
\end{align*}
\]

\[
\therefore \quad f(x) = \frac{x}{r^2 \sqrt{r^2 - x^2}}.
\]
3. (13 points) The velocity of a particle is given by \( v(t) = \sin^3(\pi t) \) ft/sec where \( t \) is in seconds.

(a) (7 points) Assume the initial position of the particle is \( x(0) = 0 \) ft.

Find the function \( s(t) \) for the position of the particle at time \( t \).

\[
\begin{align*}
S(t) &= \int \sin^3(\pi t) \, dt \\
&= \int (1 - \cos^2(\pi t) \sin(\pi t)) \, dt \\
&= -\frac{1}{\pi} \int 1 - u^2 \, du \\
&= \frac{1}{\pi} \left( \frac{u^3}{3} - u \right) + C = \frac{1}{\pi} \left( \frac{\cos^3(\pi t)}{3} - \cos(\pi t) \right) + C
\end{align*}
\]

\[S(0) = 0 \iff 0 = \frac{1}{\pi} \left( \frac{1}{3} - 1 \right) + C \iff C = \frac{2}{3\pi} \]

\[
S(t) = \frac{1}{\pi} \left( \frac{\cos^3(\pi t)}{3} - \cos(\pi t) \right) + \frac{2}{3\pi}
\]

(b) (6 points) Find the total distance traveled by the particle from \( t = 0 \) to \( t = \frac{3}{2} \) seconds.

We want:

\[
\int_0^{3/2} \left| v(t) \right| \, dt = \int_0^{3/2} \left| \sin^3(\pi t) \right| \, dt
\]

Over the interval \([0, 3/2]\):

\[
\sin^3(\pi t) = 0 \quad \text{at} \quad t = 1
\]

\[t \in \mathbb{R} \quad \text{on} \quad [0, 1] \quad \text{and} \quad \in \mathbb{R} \quad \text{on} \quad [1, 3/2] \]

So we get:

\[
\int_0^{3/2} \left| \sin^3(\pi t) \right| \, dt = \int_0^1 \sin^3(\pi t) \, dt + \int_{1}^{3/2} \sin^3(\pi t) \, dt
\]

Using the antiderivative found above:

\[
\begin{align*}
\cos(\pi) &= -1 \\
\cos \left( \frac{3\pi}{2} \right) &= 0
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{\pi} \left[ \frac{\cos^3(\pi t)}{3} - \cos(\pi t) \right] \bigg|_0^1 - \frac{1}{\pi} \left[ \frac{\cos^3(\pi t)}{3} - \cos(\pi t) \right] \bigg|_0^{3/2} \\
&= \frac{1}{\pi} \left[ \left( \frac{1}{3} + 1 \right) - \left( \frac{1}{3} - 1 \right) \right] - \frac{1}{\pi} \left[ \left( 0 - 0 \right) - \left( \frac{1}{3} - 1 \right) \right] \\
&= \frac{1}{\pi} \left( \frac{4}{3} \right) - \frac{1}{\pi} \left( \frac{-2}{3} \right) = \frac{6}{3\pi} = \boxed{\frac{2}{\pi} \text{ feet}}
\end{align*}
\]
4. (14 points) Let $R$ be the region enclosed by: the $x$-axis, the line $y = 5$, the line $x = -2$, and the portion of the curve $y = 5 \tan(x)$ between $x = 0$ and $x = \pi/4$. The region $R$ is rotated around the line $x = -2$ to form a solid of revolution. The units are meters. In parts (b) and (c) take $g$ to be 9.8 m/sec$^2$ and take the density of water to be 1000 kg/m$^3$.

Write each of the following in terms of integrals, but do **not** evaluate the integrals.

(a) (7 points) the volume of the resulting container;

\[
V = \int_{0}^{5} \pi \left( \text{radius} \right)^2 \, dy = \int_{0}^{5} \pi \left( \arctan \left( \frac{y}{5} \right) + 2 \right)^2 \, dy
\]

(b) (4 points) the amount of work (in Joules) required to empty the container of water, if water is filled up to the level of 3 meters, and there's an outtake pipe at height 4 meters;

Slice the water (interval $[0, 3]$) into $n$ horizontal slices. The $i^{th}$ slice is a disk of thickness $dy$ and we have:

- Weight of slice: $F_i = (9.8)(1000) \pi \left( 2 + \tan^{-1} \left( \frac{y_i}{5} \right) \right)^2 \, dy$
- Distance to lift slice: $d_i = 4 - y_i$

Work $W = \lim_{n \to \infty} \sum_{i=1}^{n} 9800 \pi \left( 2 + \tan^{-1} \left( \frac{y_i}{5} \right) \right)^2 (4 - y_i) \, dy$

\[
W = \int_{0}^{3} 9800 \pi \left( 2 + \tan^{-1} \left( \frac{y}{5} \right) \right)^2 (4 - y) \, dy
\]

(c) (3 points) the amount of work (in Joules) required to empty the container of water if the container is filled to the top with water and the outtake pipe is at height 7 meters (above the $x$-axis).
5. (10 points) Find the coordinates \((\bar{x}, \bar{y})\) for the center of mass of the region shown below.

By symmetry: \(\bar{x} = 0\)

Note: We will compute \(\bar{y}\) for the right half of the region since, by symmetry again, it will be the same as \(\bar{y}\) for the entire region.

There are many ways to compute \(\bar{y}\). Here are a few:

**Method I:** \(\bar{y} = \frac{1}{\text{area}} \text{M}_y = \frac{1}{\frac{7}{6}} \left[ \int_0^2 \frac{1}{2} (x)^2 \, dx + \int_{3}^4 \frac{1}{2} (-2x+8)^2 \, dx \right] = \frac{1}{\frac{7}{6}} \left[ \frac{20}{21} \right] = \frac{20}{21}\)

**Method II:** Switch the axes and compute \(\bar{x}\) for the resulting region:

original \(\bar{x} = \text{new } \bar{x} = \frac{1}{\text{area}} \int_0^2 x \cdot f(x) \, dx = \frac{1}{\frac{7}{6}} \int_0^2 x (-\frac{1}{2}x + 4) \, dx = \frac{20}{21}\)

**Method III:** Decompose the region into \(A_1\) & \(A_2\) and use

\[\bar{y} = \frac{\text{A}_1 \bar{y}_1 + \text{A}_2 \bar{y}_2}{\text{A}_1 + \text{A}_2} = \frac{(6)(1) + (1) \left[ \int_3^4 \frac{1}{2} (-2x+8)^2 \, dx \right]}{6 + 1} = \frac{\frac{20}{3}}{7} = \frac{20}{21}\]

Either way, the answer is:

\[\left( \bar{x}, \bar{y} \right) = \left( 0, \frac{20}{21} \right)\]
6. (10 points) Find the explicit solution \( y = y(x) \) to the initial value problem

\[
\frac{dy}{dx} = y^2 e^{\sqrt{x}}, \quad y(0) = \frac{1}{5}.
\]

Separate the variables and integrate:

\[
\int \frac{1}{y} dy = \int e^{\sqrt{x}} dx
\]

Rationalizing substitution

\[ u = \sqrt{x} \Rightarrow x = u^2 \]

\[ dx = 2u du \]

Integration by Parts

\[ w = u \quad dv = e^u du \]

\[ dw = du \quad v = e^u \]

\[
-\frac{1}{y} = 2ue^u - 2e^u + C
\]

\[
-\frac{1}{y} = 2\sqrt{x} e^{\sqrt{x}} - 2e - 3 + C
\]

\( y(0) = \frac{1}{5} \Rightarrow -5 = 0 - 2 + C \Rightarrow C = -3 \)

\[
\therefore \quad \frac{1}{y} = 2\sqrt{x} e^{\sqrt{x}} - 2e - 3
\]

\[
y = \frac{-1}{2\sqrt{x} e^{\sqrt{x}} - 2e - 3} = \frac{1}{3 + 2e^{\sqrt{x}} - 2\sqrt{x} e^{\sqrt{x}}}
\]
7. (13 points) Suppose you drop a stone of mass \( m \) from a great height in the earth's atmosphere, and the only forces acting on the stone are the earth's gravitational attraction and a retarding force due to air resistance, which is proportional to the velocity \( v \). Take downward to be the positive direction. Then, since \( F = ma \) and \( a = \frac{dv}{dt} \), we have the differential equation:

\[
\frac{dv}{dt} = mg - kv,
\]

where \( k \) is a positive constant. Suppose that the mass is \( m = 1 \) kg, and take \( g = 9.8 \) m/sec\(^2\).

(a) (6 points) Solve the differential equation to find a formula for \( v(t) \). Your answer will involve \( k \).

With \( m = 1 \) kg and \( g = 9.8 \), we have:

\[
\frac{dv}{dt} = 9.8 - kv
\]

Separating the variables and integrating:

\[
\int \frac{1}{9.8 - kv} \, dv = \int \frac{1}{dt}
\]

\[
-\frac{1}{k} \ln |9.8 - kv| = t + C
\]

Initial condition is: \( v(0) = 0 \ ("\text{drop}") \), so \( -\frac{1}{k} \ln 9.8 = C \)

Replacing in the equation:

\[
-\frac{1}{k} \ln |9.8 - kv| = t - \frac{1}{k} \ln 9.8
\]

Solving for \( v \):

\[
|9.8 - kv| = e^{kt} \cdot e^{\frac{1}{k} \ln 9.8}
\]

\[
9.8 - kv = \pm 9.8e^{-kt}
\]

Using the initial condition \( v(0) = 0 \), we get:

\[
v = \frac{9.8}{k} \left(1 - e^{-kt}\right)
\]

(b) (3 points) Compute the terminal velocity \( v_\infty \) (the limiting velocity as \( t \to \infty \)).

Your answer will involve the positive constant \( k \).

\[
v_\infty = \lim_{t \to \infty} v = \frac{9.8}{k} \left(1 - e^{-kt}\right) = \frac{9.8}{k} (1 - 0)
\]

\[
= \frac{9.8}{k} \text{ m/s}.
\]

(c) (4 points) If \( v_\infty = 70 \) m/sec, find the speed of the stone after 3 sec.

\[
70 = \frac{9.8}{k} \Rightarrow k = \frac{9.8}{70} = 0.14
\]

From (a):

\[
v = \frac{9.8}{k} \left(1 - e^{-kt}\right) = 70 \left(1 - e^{-\frac{9.8}{70} \cdot 3}\right)
\]

At \( t = 3 \) sec:

\[
v = 70 \left(1 - e^{-\frac{9.8}{70} \cdot 3}\right) = 70 \left(1 - e^{-0.42}\right) \approx 24 \text{ m/s}
\]
8. (12 points) Suppose that the graph of $f$ is as shown:

(a) (4 points) Compute the average value of this function over the interval $[0, 10]$.

\[
\bar{f} = \frac{1}{10} \int_0^{10} f(x) \, dx = \frac{1}{10} \left( \text{signed area between } y = f(x) \text{ and the x-axis from } x = 0 \text{ to } x = 10 \right)
\]

\[
= \frac{1}{10} \left( 3 + 5 + 7 - \frac{25}{2} + \frac{25}{2} \right)
\]

\[
= \frac{1}{10} \left( 10 \right) = 1
\]

(b) Define a new function $A(x) = \int_x^8 f(t) \, dt$, where $f$ is the same function as above.

i. (2 points) Compute $A(2)$.

\[
A(2) = \int_2^8 f(t) \, dt = 5 + 5 - 3 = 7
\]

ii. (6 points) Compute $A'(2)$.

\[
A'(x) = \frac{d}{dx} \int_x^8 f(t) \, dt = \frac{d}{dx} \left( \int_x^8 f(t) \, dt \right) = \frac{d}{dx} \left( \int_0^8 f(t) \, dt - \int_0^x f(t) \, dt \right)
\]

\[
= \frac{d}{dx} \left( - \int_0^x f(t) \, dt \right) + \frac{d}{dx} \left( \int_0^8 f(t) \, dt \right)
\]

\[
= -f(x) + \left( \int_0^8 f(t) \, dt \right) \text{ (Reverse Chain Rule)}
\]

\[
= -f(x) + \int_0^8 f(t) \, dt
\]

\[
\therefore A'(x) = -f(x) + 3x^2 + (x^3)
\]

\[
A'(2) = -f(2) + 12 f(8) \text{ (Reading y-values on graph)}
\]

\[
= -2 + 12 \cdot 5 = 58
\]