1. (10 total points) Evaluate the following definite integrals. Simplify and box your answers.

(a) (5 points) $\int_0^{\pi/4} \tan^2\theta \sec^4\theta \, d\theta$

$$= \int_0^{\pi/4} \tan^2\theta \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta (\tan^2\theta + 1) \sec^2\theta \, d\theta$$

$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$$= \int_0^1 u^2(u^2+1)\,du = \int_0^1 u^4 + u^2 \, du$$

$$= \frac{1}{5}u^5 + \frac{1}{3}u^3 \Big|_0^1 = \frac{1}{5} + \frac{1}{3} = \boxed{\frac{8}{15}}$$

(b) (5 points) $\int_0^1 x^2 \arcsin(x)\,dx$

IBP: $u = \arcsin x$ $\quad dv = x^2\,dx$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \arcsin x \Big|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx$$

u-sub: $u = 1-x^2$
$du = -2x\,dx$
(or trig sub $u = \sin\theta$)

$$= \left(\frac{1}{3}(1)\left(\frac{\pi}{2}\right) - 0\right) - \frac{1}{3}\int_1^0 \frac{(1-u)}{\sqrt{u}}\left(-\frac{1}{2}\right)du$$

$$= \frac{\pi}{6} - \frac{1}{6}\int_0^1 u^{-1/2} - u^{1/2}\,du$$

$$= \frac{\pi}{6} - \frac{1}{6}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{\pi}{6} - \frac{1}{6}\left(2 - \frac{2}{3}\right)$$

$$= \boxed{\frac{\pi}{6} - \frac{2}{9}}$$

2. (10 points) Evaluate the following indefinite integrals.

(a) (5 points) $\int \sqrt{10x - x^2}\, dx$

complete the square

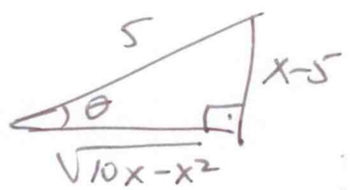
$$= \int \sqrt{25 - (x-5)^2}\, dx$$

trig sub: $x - 5 = 5 \sin\theta$
$dx = 5 \cos\theta \, d\theta$

$$= \int \sqrt{25 \cos^2\theta}\; 5 \cos\theta \, d\theta$$

$$= 25 \int \cos^2\theta \, d\theta = \frac{25}{2} \int 1 + \cos(2\theta)\, d\theta$$

$$= \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C = \frac{25}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$



$$= \frac{25}{2}\left[\arcsin\left(\frac{x-5}{5}\right) + \frac{(x-5)\sqrt{10x-x^2}}{25}\right] + C$$

$$= \boxed{\frac{25}{2}\arcsin\left(\frac{x-5}{5}\right) + \frac{1}{2}(x-5)\sqrt{10x-x^2} + C}$$

$\sin\theta = \frac{x-5}{5}$

$\cos\theta = \frac{\sqrt{10x-x^2}}{5}$

(b) (5 points) $\int \frac{x-1}{x^3+x}\, dx$

$$\frac{x-1}{x^3+x} = \frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x - 1 = (A+B)x^2 + Cx + A \implies A = -1,\ C = 1,\ B = -A = 1$$

$$\int \frac{x-1}{x^3+x}\, dx = \int \frac{-1}{x} + \frac{x+1}{x^2+1}\, dx$$

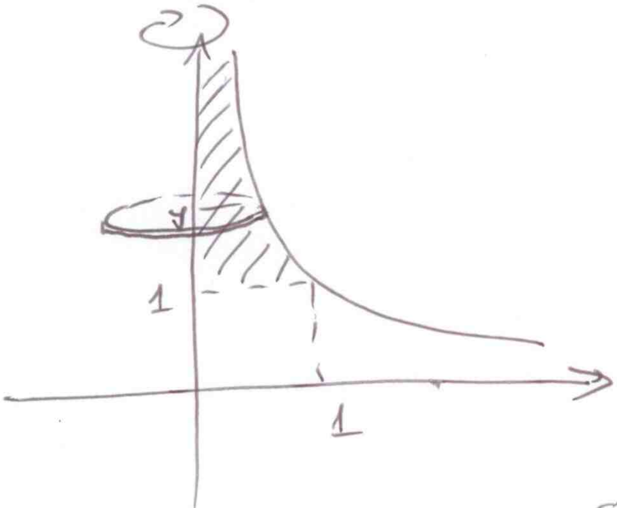
$$= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\, dx$$

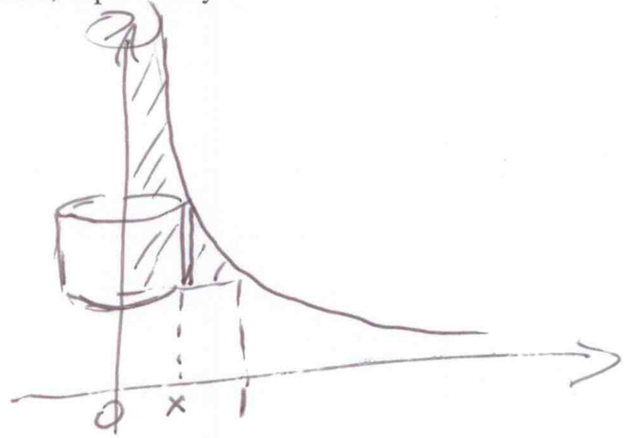
$$= \boxed{-\ln|x| + \frac{1}{2}\ln(x^2+1) + \arctan(x) + C}$$

3. (10 points) Consider the infinite region in the first quadrant of the xy-plane, above the line $y = 1$, and to the left of the curve

$$y = \frac{1}{\sqrt{x}}$$

Rotate this region about the y-axis, and determine whether the volume of the resulting solid is finite or infinite. If it is finite, find the volume. If it is infinite, explain why.





Using Disks:

$$V = \int_1^\infty \pi \left(\frac{1}{y^2}\right)^2 dy$$

$$= \lim_{t\to\infty} \int_1^t \pi y^{-4} dy$$

$$= \pi \lim_{t\to\infty} \left(-\frac{1}{3y^3}\right)\Big|_1^t$$

$$= \pi \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3}\right)$$

$$= \pi \left(0 + \frac{1}{3}\right)$$

$$= \boxed{\frac{\pi}{3}}$$

OR Using Shells:

$$V = \int_0^1 2\pi\, x \left(\frac{1}{\sqrt{x}} - 1\right) dx$$

$$= 2\pi \int_0^1 \sqrt{x} - x \, dx$$

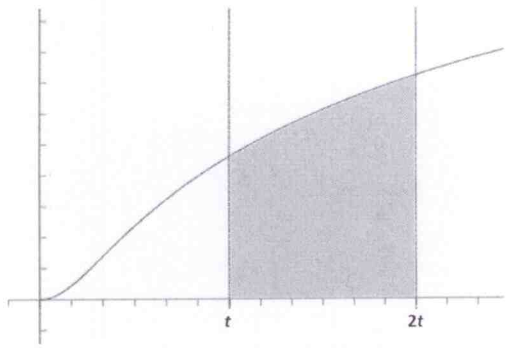
$$= 2\pi \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right)\Big|_0^1$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= 2\pi \frac{1}{6}$$

$$= \boxed{\frac{\pi}{3}}$$

4. (10 total points) The figure on the right shows a region bounded to the left by the line $x = t$, to the right by $x = 2t$, on the top by the curve $y = \ln(x^2 + 1)$, and on the bottom by the x-axis.



(a) (2 points) Set up an integral for the area $A(t)$ of this region. DO NOT EVALUATE the integral.

$$A_{(t)} = \int_t^{2t} \ln(x^2+1)\,dx$$

(b) (4 points) Compute $A'(1)$

$$A_{(t)} = -\int_0^t \ln(x^2+1)\,dx + \int_0^{2t} \ln(x^2+1)\,dx$$

Applying FTC I (& Chain Rule):

$$A'(t) = -\ln(t^2+1) + \ln(4t^2+1)\cdot 2$$

Evaluating at $t=1$

$$A'(1) = \boxed{-\ln(2) + 2\ln 5} = \boxed{\ln\left(\frac{25}{2}\right)}$$

(c) (4 points) Set up an integral for the arc length $L(t)$ of the top boundary of this region (that is, the arc length of the curve $y = \ln(x^2 + 1)$, $t \le x \le 2t$). DO NOT EVALUATE the integral.

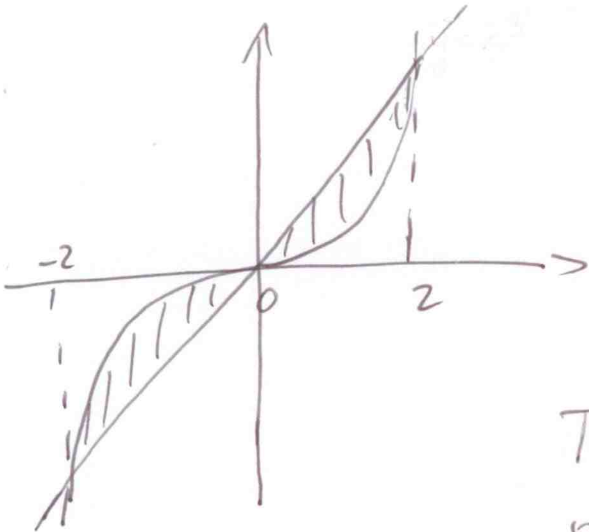
$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

$$\boxed{L = \int_t^{2t} \sqrt{1 + \left(\frac{2x}{x^2+1}\right)^2}\,dx}$$

5. (10 points) The curves:

$$y = x^3 \qquad \text{and} \qquad y = 4x$$

enclose two regions in the plane. Find the total area of these regions.



Intersection points:

$x^3 = 4x$

$x(x^2-4) = 0$

$x = 0,\ x = \pm 2$

$$\text{Total area} = \int_{-2}^{2} |4x - x^3|\,dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx + \int_{0}^{2} (4x - x^3)\,dx$$

$$= 2\int_{0}^{2} (4x - x^3)\,dx$$

$$= 2\left[2x^2 - \tfrac{1}{4}x^4\right]\Big|_0^2$$

$$= 2\,[8-4] = \boxed{8}$$