Worksheet for Week 1: Circles and lines

This worksheet is a review of circles and lines, and will give you some practice with algebra and with graphing. Also, this worksheet introduces the idea of “tangent lines” to circles. Later on in Math 124, you’ll learn how to find tangent lines to many other types of curves.

1. Two circles, called $C_1$ and $C_2$, are graphed below. The center of $C_1$ is at the origin, and the center of $C_2$ is the point in the first quadrant where the line $y = x$ intersects $C_1$. Suppose $C_1$ has radius 2. $C_2$ touches the $x$ and $y$ axes each in one point. What are the equations of the two circles?

![Diagram of two circles](image)

**Solution:** The equation for a circle with center $(h, k)$ and radius $r$ is

$$(x - h)^2 + (y - k)^2 = r^2.$$ 

The circle $C_1$ is centered at $(0, 0)$ and has radius 2, so its equation is $x^2 + y^2 = 4$. $C_2$’s center is at the point where the line $y = x$ meets $C_1$. So we need to solve for $x$ and $y$ provided that the two equations

$$x^2 + y^2 = 4 \quad \quad x = y$$

are true. Substituting $x = y$ into the left equation and solving for $x$ and $y$, we get two solutions: $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Since the point we want is in the first quadrant, we know the center of $C_2$ must be $(\sqrt{2}, \sqrt{2})$. We know $C_2$ just touches the $x$ and $y$ axes in one point, so the radius of $C_2$ is $\sqrt{2}$. Finally, the equation of $C_2$ is

$$(x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 2.$$
2. Let $C$ be the circle of radius 5 centered at the origin. The tangent line to $C$ at a point $Q$ is the line through $Q$ that’s perpendicular to the radial line connecting $Q$ to the center. (See picture.) Use this information to find the equations of the tangent lines at $P$ and $Q$ below.

![Diagram of a circle with points P and Q and tangent lines]

**Note:** Later in Math 124, you’ll learn how to find tangent lines to curves that are not circles!

**Solution:** If one line has slope $m$ and another line has slope $n$, then the lines are perpendicular if $m = -\frac{1}{n}$.

The radial line through $(0, 0)$ and $Q = (3, 4)$ has slope $\frac{4}{3}$. So the tangent line must have slope $-\frac{3}{4}$. Now use the point-slope formula to get the equation of the line through $(3, 4)$ with slope $-\frac{3}{4}$:

$$y = -\frac{3}{4}x + \frac{25}{4}.$$  

Similarly, the radial line through $(0, 0)$ and $P = (-4, 3)$ has slope $-\frac{3}{4}$. So the tangent line through $P$ has slope $\frac{4}{3}$. Its equation is

$$y = \frac{4}{3}x + \frac{25}{3}.$$
3. Sketch the circle of radius 2 centered at \((3, -3)\) and the line \(L\) with equation \(y = 2x + 2\). Find the coordinates of all the points on the circle where the tangent line is perpendicular to \(L\).

**Solution:** The line \(L\) has slope 2, so we want to find tangent lines to the circle with slope \(-\frac{1}{2}\). We can do this by finding points \(P\) on the circle so that the radial line from \((3, -3)\) to \(P\) has slope 2. In other words, we want points \((x, y)\) so that the following two equations are true:

\[
(x - 3)^2 + (y + 3)^2 = 4 \quad \frac{y + 3}{x - 3} = 2.
\]

Simplifying, the second equation becomes \(y = 2x - 9\). Plug this in to the first equation and solve for \(x\) and \(y\). We get two points:

\[
\left(3 + \frac{2}{\sqrt{5}}, -3 + \frac{4}{\sqrt{5}}\right) \quad \text{and} \quad \left(3 - \frac{2}{\sqrt{5}}, -3 - \frac{4}{\sqrt{5}}\right).
\]
4. Draw the circle with equation \( x^2 + y^2 = 25 \) and the points \( P = (-3, -4) \) and \( Q = (-8, 0) \). Explain why \( P \) is on the circle. Is the line through \( P \) and \( Q \) tangent to the circle? How do you know?

**Solution:**

This circle is centered at the origin and has radius 5. The point \( P \) is on the circle because its coordinates satisfy the circle’s equation: \((-3)^2 + (-4)^2 = 25\). Draw the line through \( Q \) and \( P \). It looks tangent to the circle, but we need to check with the definition in Question 2. The radial line from the center of the circle to \( P \) has slope \( \frac{4}{3} \), so the tangent line at \( P \) must have slope \( -\frac{3}{4} \). But the line through \( P \) and \( Q \) has slope \( -\frac{4}{5} \), so it actually is not tangent to the circle.