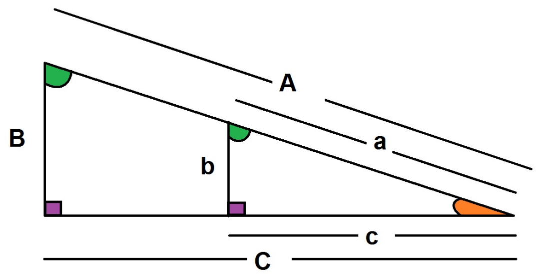
# Worksheet for Week 7: Geometry Review

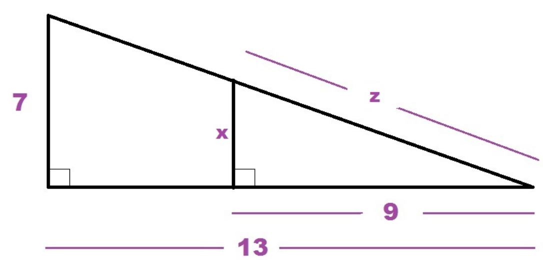
Many of the story problems will require writing down functions using **geometry**. You need **area** and **volume** equations, the **Pythagorean Theorem**, definitions of the **trigonometric functions** on the right triangle and **similar triangles**. This worksheet covers geometry of triangles.

1. **Using Similar Triangles**
   Similar triangles have a common ratio for their corresponding sides, opposite the angles with the same measures. In particular, when one right triangle sits inside another, the two triangles are similar so you get the ratios:

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$




(a) Use the fact that two right triangles are similar to compute $x$ and $z$.




**Solution:**

$$\frac{7}{x} = \frac{13}{9}$$

So $x = \dfrac{63}{13}$. For $z$, use either $z^2 = 9^2 + x^2$ or

$$\frac{\sqrt{13^2 + 7^2}}{z} = \frac{13}{9}.$$

In any case, $z = \dfrac{9\sqrt{218}}{13}$.

(b) Use the fact that two right triangles are similar to compute $y$ and $z$ in terms of $x$.

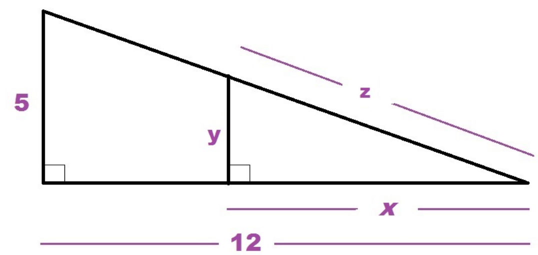
**Solution:**

$$\frac{y}{5} = \frac{x}{12}$$

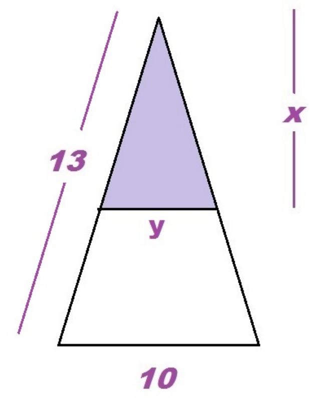
So $y = \dfrac{5x}{12}$. For $z$, use either $z^2 = x^2 + y^2$ or

$$\frac{\sqrt{12^2 + 5^2}}{z} = \frac{12}{x}.$$

In any case, $z = \dfrac{13x}{12}$.



(c) Find $y$ in terms of $x$. You need right triangles to get started on this one.

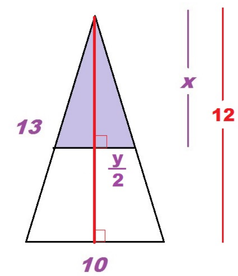



**Solution:**

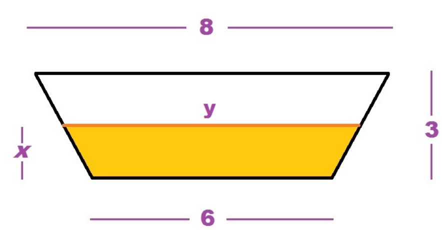
The height of the triangle is $\sqrt{13^2+5^2} = 12$. Using similar triangles

$$\frac{x}{12} = \frac{\frac{y}{2}}{\frac{10}{2}}$$

so $y = \dfrac{5x}{6}$.



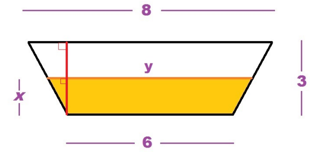

(d) Find the area of the shaded region in terms of $x$. You need right triangles to get started on this one, too.



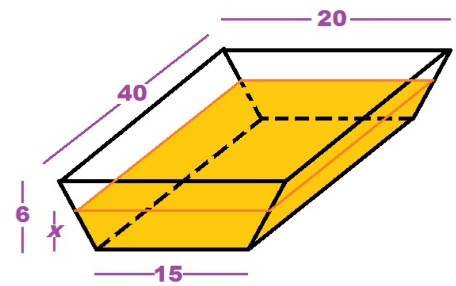

**Solution:**
Dropping a height (shown in red) we see two similar right triangles:

$$\frac{3}{x} = \frac{\frac{8-6}{2}}{\frac{y-6}{2}}$$

so $y = \dfrac{2x}{3} + 6$. So the area of the (yellow) trapezoid is $\left(\dfrac{y+6}{2}\right)x = \dfrac{x^2}{3} + 6x$.



(e) The tank shown on the right is filled with water to a depth $x$. Find the volume of water in the tank in terms of $x$ and the total volume of the tank. Remember to compute the volume you multiply base area by height. Which one is the base?




**Solution:**
Using similar triangles

$$\frac{6}{x} = \frac{\frac{20-15}{2}}{\frac{y-15}{2}}$$

so $y = \dfrac{5x}{6} + 15$. Area of the (shaded) trapezoid is the average of the opposite sides multiplied by the height:

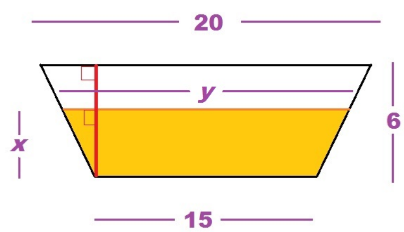
$$\left[\frac{\left(\frac{5x}{6}+15\right)+15}{2}\right] \cdot x = \frac{5}{12}x^2 + 15x.$$

So the volume of water is

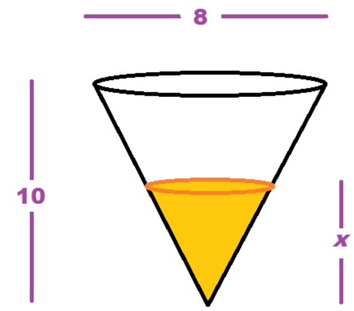
$$V(x) = \left(\frac{5}{12}x^2 + 15x\right) \cdot 40 = \frac{50}{3}x^2 + 600x$$

The total volume of the tank is $V(6) = 4200$ cubic units.

Here is the cross-section of the tank. Again, we drop a perpendicular to form two similar right triangles:



(f) The tank shown on the right is filled with water to a depth $x$. Find the volume of water in the tank in terms of $x$ and the total volume of the tank. What is the formula for the volume of a cone?




**Solution:**
Using similar triangles

$$\frac{10}{x} = \frac{4}{r}$$

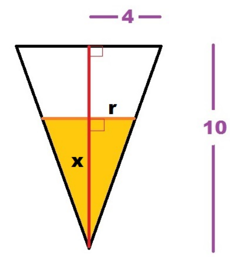
so $r = \dfrac{2x}{5}$. Then,

$$V(x) = \frac{1}{3}\pi\left(\frac{2x}{5}\right)^2 \cdot x = \frac{4\pi}{75}x^3.$$

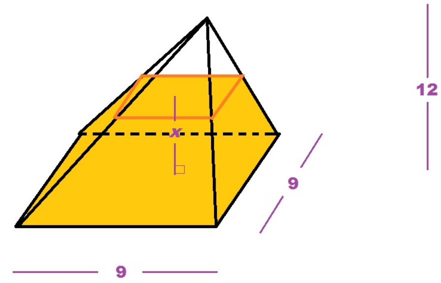
The total volume of the tank is

$$V(10) = \frac{1}{3}\pi\left(\frac{20}{5}\right)^2 \cdot 10 = \frac{160\pi}{3}.$$

Here is a cross-section:



(g) The tank shown on the right is filled with water to a depth $x$. Find the total volume of the tank and the volume of water in the tank in terms of $x$. What is the formula for the volume of a pyramid?

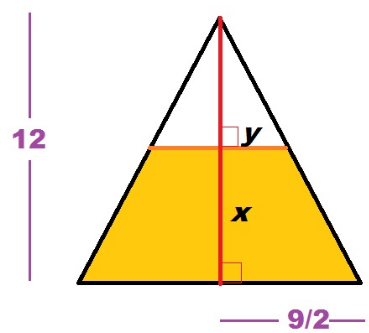



**Solution:**

The volume of the tank is

$$\frac{1}{3} \cdot 9^2 \cdot 12 = 324$$

Here is a cross-section:




Using similar triangles

$$\frac{\frac{9}{2}}{y} = \frac{12}{12 - x}$$

so $y = \dfrac{3(12 - x)}{8}$.

Then, the base area of the empty top part of the pyramid is

$$(2y)^2 = \left(\frac{3(12 - x)}{4}\right)^2$$

so the volume of the empty top part of the pyramid is

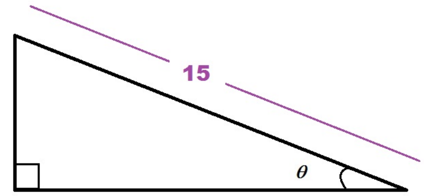
$$V(x) = \frac{1}{3}\left(\frac{3(12 - x)}{4}\right)^2 \cdot (12 - x) = \frac{3}{16}(12 - x)^3$$

and the volume of the water is

$$V(x) = 324 - \frac{3}{16}(12 - x)^3.$$

2. **Circles and Angles -** For this question you will need the trigonometric functions. Remember that we always use RADIANS for angles in calculus.

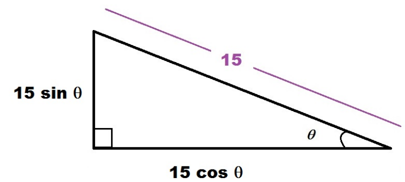
   (a) Find the area and the perimeter of the triangle.




**Solution:**

$$\text{Area} = \frac{225 \sin\theta \cos\theta}{2}$$

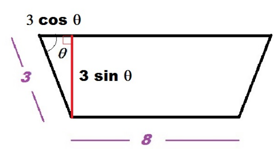
$$\text{Perimeter} = 15\,(1 + \sin\theta + \cos\theta)$$




   (b) Find the area and the perimeter of the trapezoid.


**Solution:**

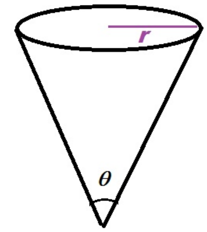
Area of a trapezoid is the average of its parallel sides multiplied by its height (you can verify this by taking a trapezoid and dividing it into two triangles and a rectangle) so



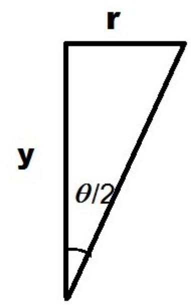

$$\text{Area} = \left(\frac{8 + (8 + 6\cos\theta)}{2}\right) \cdot 3\sin\theta$$

$$\text{Perimeter} = 22 + 6\cos\theta$$

(c) Find the volume of the cone in terms of $r$ and $\theta$.



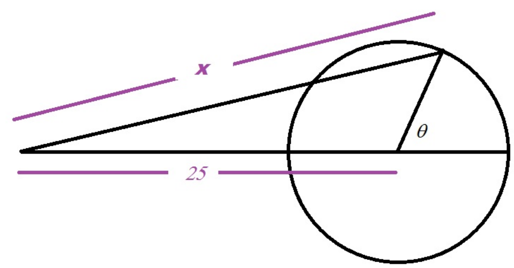

**Solution:**



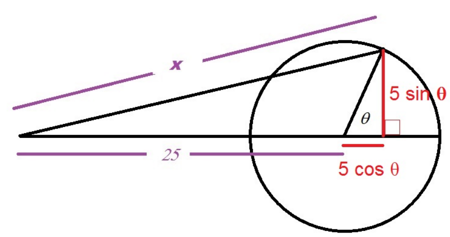

The volume of a cone is one third height times base area:

$$V = \frac{1}{3}r^2 \cdot y = \frac{\pi r^3}{3\tan\left(\frac{\theta}{2}\right)}$$

(d) Find $x$ given that the radius of the circle is 5 units long.




**Solution:**
After dropping a perpendicular to form a right triangle we use the Pythagorean Theorem




$$x^2 = (5\sin\theta)^2 + (25 + 5\cos\theta)^2$$

so after taking the square root and simplifying

$$x = \sqrt{650 + 250\cos\theta}.$$

You can also use the **Law of Cosines** for this problem. Look it up.