1. (10 points) Compute the derivatives of the following functions. Do **not** simplify your answers.

(a) $y = 3^x \cdot (\tan(x))^2$

$$\frac{dy}{dx} = (3^x)'(\tan^2 x) + 3^x(\tan^2 x)' \quad \text{(Product Rule)}$$

$\downarrow$ Exp Rule $\qquad\qquad$ $\downarrow$ Chain Rule

$$= \boxed{(3^x \ln 3)(\tan^2 x) + 3^x(2\tan x \sec^2 x)}$$

$$= 3^x \tan x(\ln 3 \tan x + 2\sec^2 x)$$

(b) $y = \sqrt{x + \sqrt{x^2+1}}$ $\qquad$ Double Chain Rule:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x^2+1}}}\left(x + \sqrt{x^2+1}\right)'$$

$$= \boxed{\frac{1}{2\sqrt{x+\sqrt{x^2+1}}}\left(1 + \frac{1}{2\sqrt{x^2+1}}(2x)\right)}$$

OR

$$= \boxed{\frac{1}{2}\left(x+\sqrt{x^2+1}\right)^{-1/2}\left(1 + \frac{1}{2}(x^2+1)^{-1/2}(2x)\right)}$$

2. Consider the curve implicitly defined by the equation:

$$y^2 - 2e^{xy} = x$$

(a) (6 points) Compute $\frac{dy}{dx}$ in terms of x and y.

$$\underbrace{\frac{d}{dx}(y^2)} - 2\underbrace{\frac{d}{dx}(e^{xy})} = \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} - 2e^{xy}\frac{d}{dx}(xy) = 1$$

$$2y\frac{dy}{dx} - 2e^{xy}\left(y + x\frac{dy}{dx}\right) = 1$$

Solving for $\frac{dy}{dx}$: $\underline{2y\frac{dy}{dx}} - 2ye^{xy} - \underline{2xe^{xy}\frac{dy}{dx}} = 1$

$$(2y - 2xe^{xy})\frac{dy}{dx} = 1 + 2ye^{xy}$$

$$\boxed{\frac{dy}{dx} = \frac{1 + 2ye^{xy}}{2y - 2xe^{xy}}}$$

(b) (4 points) Find the x-intercept of this curve, and the equation of the tangent line at the x-intercept.

x-intercept: $y=0 \Rightarrow (0)^2 - 2e^0 = x \Rightarrow x = -2$

So the x-intercept is the point $(x,y) = (-2,0)$

slope at that point is $\left.\frac{dy}{dx}\right|_{(-2,0)} = \frac{1+2(0)e^0}{2(0)-2(-2)e^0} = \frac{1}{4}$

tan line: $y = \frac{1}{4}(x-(-2)) + 0$

so $\boxed{y = \frac{1}{4}(x+2)}$ OR $\boxed{y = \frac{1}{4}x + \frac{1}{2}}$

3. (9 points) Compute the derivative $\frac{dy}{dx}$ in terms of x, for the following function (with domain $x > 0$):

$$y = \left(\frac{x}{x+2}\right)^x$$

Show all steps clearly. Simplify and box your final answer.

Logarithmic differentiation:

$$\ln(y) = x \ln\left(\frac{x}{x+2}\right)$$

$$\frac{1}{y} y' = 1 \cdot \ln\left(\frac{x}{x+2}\right) + x\left(\frac{1}{\left(\frac{x}{x+2}\right)} \cdot \left(\frac{x}{x+2}\right)'\right)$$

$$= \ln\left(\frac{x}{x+2}\right) + x \frac{\cancel{x+2}}{\cancel{x}} \cdot \frac{(\cancel{x}+2)-\cancel{x}}{(x+2)^{\cancel{2}}}$$

$$\frac{1}{y} y' = \ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}$$

$$y' = y\left(\ln \frac{x}{x+2} + \frac{2}{x+2}\right)$$

$$\boxed{y' = \left(\frac{x}{x+2}\right)^x \left[\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right]}$$

OR

$$\ln y = x(\ln x - \ln(x+2))$$

$$\frac{1}{y} y' = (\ln x - \ln(x+2)) + x\left(\frac{1}{x} - \frac{1}{x+2}\right)$$

$$y' = y\left[\ln x - \ln(x+2) + 1 - \frac{x}{x+2}\right]$$

$$y' = \left(\frac{x}{x+2}\right)^x \left[\underbrace{\ln x - \ln(x+2)} + \underbrace{1 - \frac{x}{x+2}}\right]$$

$$= \left(\frac{x}{x+2}\right)^x \left[\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right]$$

OR: Correctly write y as an exponential with base e, then differentiate

$$y = \left(\frac{x}{x+2}\right)^x = e^{x \ln(x/x+2)}$$

$$\frac{dy}{dx} = e^{x \ln(x/x+2)} \left[x \ln\left(\frac{x}{x+2}\right)\right]'$$

$$= e^{x \ln(x/x+2)} \left[1 \cdot \ln\left(\frac{x}{x+2}\right) + x \frac{1}{\left(\frac{x}{x+2}\right)} \left(\frac{x}{x+2}\right)'\right]$$

$$= \dots$$

$$\boxed{\begin{aligned} &= e^{x \ln(x/x+2)} \left[\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right] \\ &= \left(\frac{x}{x+2}\right)^x \left[\ln\left(\frac{x}{x+2}\right) + \frac{2}{x+2}\right] \end{aligned}}$$

4. (8 points) A particle is moving along a path (curve) according to the parametric equations, for $t \geq 0$:

$$\begin{cases} x = 2\sqrt{t} - 1 \\ y = 4t - 2\sqrt{t} - 1 \end{cases}$$

(a) Compute the time(s) t, if any, when the tangent line to this curve is horizontal.

$y'(t) = 4 - \frac{2}{2\sqrt{t}} = 0$

$4 = \frac{1}{\sqrt{t}}$

$\sqrt{t} = 1/4$

$t = 1/16$

OR

$\frac{dy}{dx} = \frac{4 - \cancel{2}/\cancel{2}\sqrt{t}}{\cancel{2}\frac{1}{\cancel{2}\sqrt{t}}} = 4\sqrt{t} - 1 = 0$

$4\sqrt{t} = 1$

$\sqrt{t} = \frac{1}{4}$

$\boxed{t = \frac{1}{16}}$

(b) Eliminate the parameter to find the Cartesian equation for the path of this particle (an equation in x and y). Simplify the equation as much as possible.

$x = 2\sqrt{t} - 1 \Rightarrow 2\sqrt{t} = x + 1$

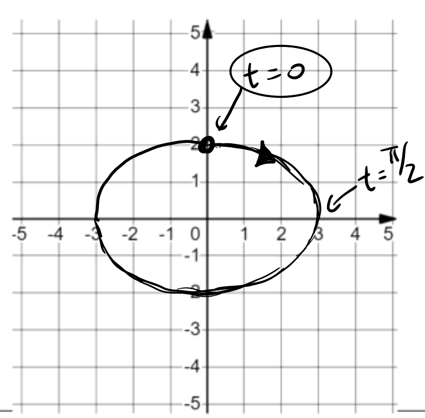
$y = (2\sqrt{t})^2 - 2\sqrt{t} - 1 = (x+1)^2 - (x+1) - 1$

$= x^2 + 2x + \cancel{1} - x - \cancel{1} - 1$

$\boxed{y = x^2 + x - 1}$

5. (4 points) Sketch the curve corresponding to the following parametric equations, for $0 \leq t \leq 2\pi$. Indicate on your picture the point at $t = 0$ and the direction of motion, as t increases:

$$x = 3\sin(t), \qquad y = 2\cos(t)$$



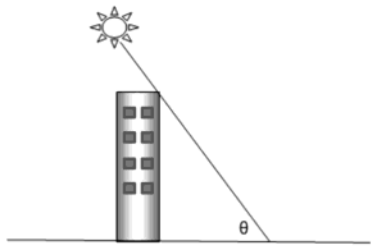
ellipse: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$t=0$ at: $(0,2)$

clockwise direction

6. (9 points) At 2 pm, the shadow cast on the ground by an 80-ft tall building is 60 ft long. At that same time, the angle θ that the sun light makes with the ground is decreasing at the rate of 0.16 radians per hour.

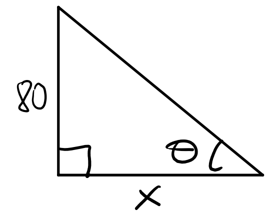
At what rate is the length of the building's shadow changing at that time?
Show all your work clearly. Box and ***include units*** *in your final answer.*



Let x = length of shadow

Know: $\frac{d\theta}{dt} = -0.16$ rad/hr.

Want: $\frac{dx}{dt} = ?$ when $x = 60$ ft



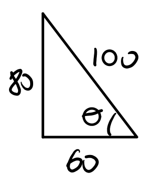
Equality relating x to θ:

$$\tan\theta = \frac{80}{x} \quad \text{or} \quad \cot\theta = \frac{x}{80}$$

Differentiate wrt time t:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-80}{x^2}\frac{dx}{dt}$$

When $x = 60$ ft



$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{100}{60}$

so: $$\left(\frac{100}{\cancel{60}}\right)^2(-0.16) = \frac{-80}{\cancel{(60)^2}}\frac{dx}{dt}.$$

$$\frac{dx}{dt} = \frac{(100)^2}{-80}(-0.16) = \frac{160\cancel{0}}{8\cancel{0}} = 20$$

Shadow is changing (increasing) at $\boxed{20 \text{ ft/hr}}$.