Midterm II Key

Tuesday, February 15, 2022 1:03 PM

Math 124 Section AA **Midterm II, February, 22th 2022** **Winter 2022**

HONOR STATEMENT
I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam.

Name

Signature

Student ID #

1.	2.	3.	4.	5.	6.	7.	8.	$\sum$
10	10	10	10	10	10	10	10	80

- You have 80 minutes for 8 problems. Check your copy of the exam for completeness.
- You are allowed to use a hand written sheet of paper (8x11 in), back and front.
- Calculators may only have basic functions, but no graphing or differentiation functions.
- Justify all your answers and show your work for credit.
- **All answers must be exact, no rounding.**

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

Problem 1. Find the derivative of

$$f(x) = \ln(\arctan(x^4)).$$

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$$f(x) = \frac{1}{\arctan(x^4)} \cdot \frac{1}{1+x^8} \cdot 4x^3$$

Problem 2. Find the derivative of

$$f(x) = \frac{\arcsin(2x) + 5x}{\cos(x^2)}.$$

$$f'(x) = \frac{\left(\frac{2}{\sqrt{1-4x^2}} + 5\right) \cdot \cos(x^2) + (\arcsin(2x) + 5x) \cdot \sin(x^2) \cdot 2x}{\cos^2(x^2)}$$

Problem 3. Find the derivative of

$$f(x) = x^3 + 3^x + x^{\sqrt{x}}.$$

$$\frac{d}{dx} x^3 = 3x^2$$

$\overline{dx}\; x = 3x$

$\frac{d}{dx} 3^x = \ln 3 \cdot 3^x$

$\ln y = \ln x^{\sqrt{x}}$ $\qquad \ln y = \sqrt{x} \cdot \ln x \quad \Big| \frac{d}{dx}$

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}$$

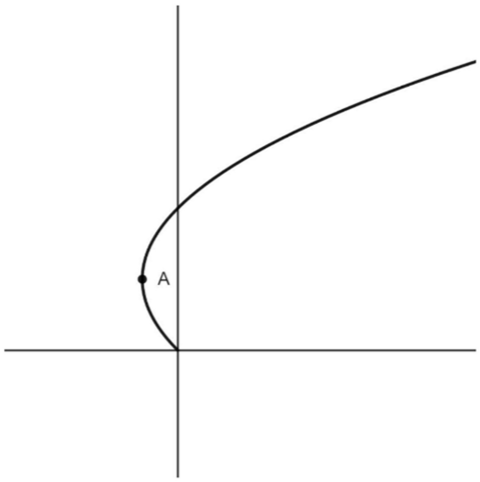
$$y' = \left(\frac{1}{2\sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}}\right) \cdot x^{\sqrt{x}}$$

$$f'(x) = 3x^2 + \ln 3 \cdot 3^x + \left(\frac{1}{2\sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}}\right) \cdot x^{\sqrt{x}}$$

Problem 4. A curve in the xy-plane is defined by the parametric equations

$$x(t) = t^4 - t^2, \quad y(t) = t^2.$$

Find the coordinates of the leftmost point of the curve. Exact values.



The leftmost point is where the tangent line is vertical

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{4t^3 - 2t} = \frac{1}{2t^2 - 1} = \infty \text{ if}$$

$$t^2 = \frac{1}{2}$$

$$t^2 = \frac{1}{2} \rightarrow \quad x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \qquad y = \frac{1}{2}$$

$$\boxed{\left(-\frac{1}{4}, \frac{1}{2}\right)}$$

Problem 5. A curve is given through the parametric equations

$x(t) = 3t^2$ [redacted] 4, $\quad y(t) = 4t^3$ [redacted] 4,

where $t > 0$. Find the equation of the tangent line to the curve that pass through the point $(0, -4)$. Exact values.

$(0,-4)$ not on curve $\rightarrow$ pt of tangency is (a,b)

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1) (a,b) on curve : $a = 3t^2-4 \quad b = 4t^3-4$

2) $\frac{\text{rise}}{\text{run}}$ = derivative : $\frac{b+4}{a} = \frac{12t^2}{6t}$

Combine: $4t^3 = 2t\cdot(3t^2-4)$

$0 = 2t^3 - 8t$

$0 = t^2(t^2-4)$

$\Rightarrow t=0$ OR $t=-2$ OR $t=2$

$t>0!$

$t=2 \rightarrow \frac{dy}{dx} = 2t\Big|_{t=2} \qquad \frac{dy}{dx} = 4$

tangent line: $y = 4x-4$

Problem 6. Given the implicitly defined curve

$8y^3 + y^2\sin(\pi(x-1)) = (x$ [redacted] $)^2,$

find the tangent line equation at the point (x, y) where $x = 1$. Exact answers.

$x=1$: $8y^3 + y^2\cdot\sin(\pi) = 1$

$y^3 = \frac{1}{8}$

$y = \frac{1}{2} \qquad \rightarrow (1, \frac{1}{2})$

derivative : $24y^2\cdot y' + 2yy'\cdot\sin(\pi(x-1)) + y^2\cdot\cos(\pi(x-1))\cdot\pi = 2x$

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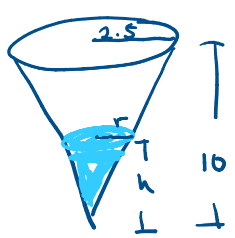
$x=1, y=\frac{1}{2}$: $24\cdot\frac{1}{4}y' + 2\cdot\frac{1}{2}y'\cdot 0 + \frac{1}{4}\cdot\pi = 2$

$$6y' = 2-\frac{\pi}{4}$$

$$y' = \frac{1}{3}-\frac{\pi}{24}$$

tangent line equation: $y = \left(\frac{1}{3}-\frac{\pi}{24}\right)(x-1)+\frac{1}{2}$

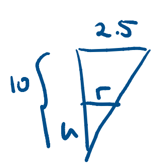
Problem 7. A conical paper cup is 5 cm in diameter at the top and 10 cm deep. Water is pouring into the cup at the rate of $2cm^3$ per second. How fast is the depth of the water in the cup rising when it is $3cm$ deep. The volume of a conical cup is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the cone.



known: $\frac{dV}{dt} = 2\,cm^3/sec$

want: $\frac{dh}{dt}$ @ $h = 3$ cm

$V = \frac{1}{3}\pi r^2\cdot h$ replace r!



$$\frac{h}{10} = \frac{r}{2.5}$$

$$V = \frac{1}{3}\pi\cdot\frac{h^3}{16}$$

$$\frac{dV}{dt} = \frac{\pi}{48}\cdot 3\cdot h^2\cdot\frac{dh}{dt}$$

$\frac{h}{10} = \frac{r}{2.5}$

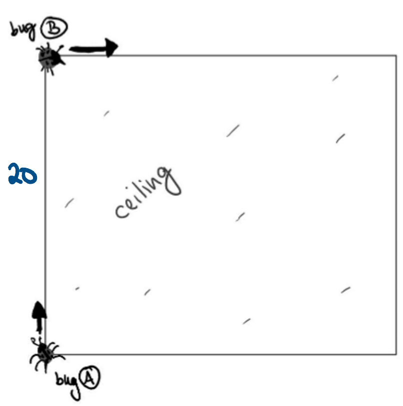
$r = \frac{h}{4}$

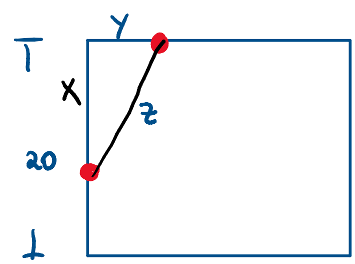
$\overline{dt} = 48 \cdots 3 \cdot h \cdot dt$

$2 \frac{cm^3}{sec} = \frac{\pi}{48} \cdot 27 \cdot \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{32}{9\pi} \ cm^3/se}$$

Problem 8. Two ladybugs are sitting in adjacent corners of the ceiling of a room, 20ft apart. They start walking along the edges of the ceiling, in directions as indicated below. If ladybug A walks at a pace of 4 feet per minute and ladybug B at a pace of 2 feet per minute, what is the rate of change of the straight line distance between them after ~~2~~ 3 minutes?





For full credit, make your own sketch, label it with all variables you use, and write out the rates that you know and that you want. Keep your answers exact.

Know: $\frac{dx}{dt} = -3 \ ft/min$

$\frac{dy}{dt} = 2 \ ft/min$

Want: $\frac{dz}{dt}$ @ 3 minutes

$$z^2 = x^2 + y^2$$

$$\cancel{2}z\frac{dz}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{10}\left(8\cdot(-4) + 6\cdot 2\right)\frac{ft}{sec}$$

$$\frac{dz}{dt} = \frac{1}{10}(-20)\,ft/sec = -2\ ft/sec$$

moment: 2 minutes:

$y = 6\ ft$

$x = 20 - 12 = 8\ ft$

$z = \sqrt{x^2+y^2} = \sqrt{64+36} = 10\ ft$