M124 Midterm 2 Key Fall 2022

Tuesday, November 8, 2022 2:55 PM

Math 124 Section **Midterm II, November 15** **Autumn 2022**

HONOR STATEMENT
I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam.

Name	Signature

Student ID #

1.	2.	3.	4.	5.	6.	7.	8.	$\sum$
10	10	10	10	10	10	10	10	80

- You have 80 minutes for 8 problems. Check your copy of the exam for completeness.
- You are allowed to use a hand written sheet of paper (8x11 in), back and front.
- Calculator : TI 30 X.
- Justify all your answers and show your work for credit.
- Some credit is given for adhering to formal aspects such as keeping the limit symbol until you take the limit, setting correct parentheses etc.
- All answers must be exact, no rounding.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

Problem 1. Find the derivative of

$$f(x) = \arctan\left(5x + \sin(x + x^2)\right) + \ln(2x^2)$$

$$f'(x) = \frac{1}{1 + [5x + \sin(x+x^2)]^2} \cdot \left(5 + \cos(x+x^2)(1+2x)\right) + \frac{4x}{2x^2}$$

no need to simplify

Problem 2. Find y' for the implicitly defined curved $x^y = y^x$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$\frac{d}{dx}$$

$$y' \ln x + \frac{y}{x} = \ln y + x \cdot \frac{y'}{y}$$

$$y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

Problem 3. Consider the function $f(x) = x^{\sin(}$[redacted]$^{\pi x)}$. Find the tangent line equation at

Problem 3. Consider the function $f(x) = x^{\sin(\text{[redacted]}\pi x)}$. Find the tangent line equation at $x = 1/2$. Show all your work and keep values exact.

$$\ln f = \sin(\pi x) \cdot \ln(x)$$

$$\frac{f'}{f} = \cos(\pi x) \cdot \pi \cdot \ln x + \sin(\pi x) \cdot \frac{1}{x}$$

$$f' = x^{\sin \pi x} \left(\cos(\pi x) \cdot \pi \cdot \ln x + \frac{\sin(\pi x)}{x} \right)$$

$$f'\left(\tfrac{1}{2}\right) = \frac{1}{2} \cdot \left(0 + \frac{1}{1/2} \right) = \frac{1}{2} \cdot 2 = 1$$

$$f\left(\tfrac{1}{2}\right) = \frac{1}{2}$$

tangent line: $y = 1\left(x - \frac{1}{2}\right) + \frac{1}{2} = x$

Problem 4. Use linearization to approximate $\sqrt{99.98}$. Round to 3 decimal places. Is it an over- or an underestimate?

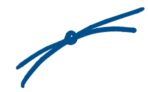
We use $f(x) = \sqrt{x}$ at $a = 100$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(100) = \frac{1}{20}$

$L(x) = \frac{1}{20}(x - 100) + 10$

$f(x) \approx L(x)$ close to 100 $\Rightarrow$

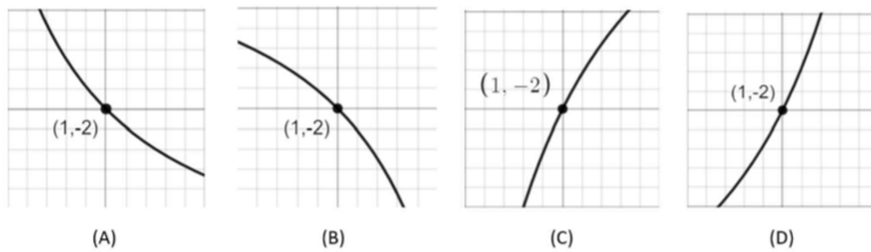
$f(99.98) \approx L(99.98) = \frac{1}{20} \cdot (-0.02) + 10 = 10 - 0.001 = \boxed{9.999}$

$f''(x) = -\frac{1}{4\sqrt{x^3}}$ $\quad f''(100) < 0$ $\Rightarrow$ concave $\downarrow$,



$\boxed{\text{overestimate}}$

Problem 5. Which of the options (A)-(D) matches the curve $x^2y^2 + xy = 2$ close to the point $(1, -2)$ best? Justify your answer.



$\frac{d}{dx}$: $\quad 2xy^2 + x^2 2y\,y' + y + xy' = 0$ $\qquad$ @ $(1,-2)$: $2\cdot 4 + 2(-2)y' - 2 + xy' = 0$

$6 = 3y' \rightarrow y' = 2$

$\frac{d^2}{(dx)^2}$: $\quad 2y^2 + 2x\cdot 2yy' + 2x\cdot 2yy' + x^2\cdot 2(y'\cdot y' + yy'') + y' + y' + xy'' = 0$

@ $(1,-2)$:

$2\cdot 4 + 4\cdot(-2)\cdot 2 + 4\cdot(-2)\cdot 2 + 2(4 - 2y'') + 4 + y'' = 0$

$8 - 16 - 16 + 8 - 4y'' + 4 + y'' = 0$

$-12 = 3y''$

$-4 = y''$ $\qquad$ concave down + increasing $\Rightarrow$ C

Problem 6. Let us consider the curve defined by the parametric equations

$$x(t) = t^4, y(t) = t^4 - t^2.$$

We assume that $t > 0$. At which point(s) on the curve has the tangent line a slope of -1? Exact values

$$\frac{dy}{dx} = \frac{4t^3 - 2t}{4t^3} = -1$$

$$\Leftrightarrow \quad 4t^3 - 2t = -4t^3$$

$$\Leftrightarrow \quad 8t^3 - t = 0 \quad \Leftrightarrow \quad t(4t^2 - 1) = 0$$

$$t = 0 \quad \text{OR} \quad t^2 = \frac{1}{4} \quad \Leftrightarrow \quad t = \pm \frac{1}{2}$$

$$t > 0 \quad \Rightarrow \quad t = \frac{1}{2}$$

$$\left(\frac{1}{16}, \frac{1}{16} - \frac{1}{4} \right) = \left(\frac{1}{16}, -\frac{3}{16} \right)$$

Problem 7. Consider the curve defined by the parametric equations

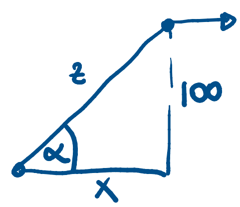
Problem 7. Consider the curve defined by the parametric equations

$$x(t) = t^3 + 1, y(t) = t^4.$$

There are two tangent lines to the curve that pass through the origin. Find the points of tangency to those tangent lines. Keep your values exact.

Problem ~~8~~ 7. A kite 100ft above the ground moves **horizontally** at a speed of 8ft/s. At what rate is the angle between the string and the horizontal changing when 200ft of string has been let out?

(a) Sketch the situation and label all relevant quantities.



(b) • KNOWN rate: $\frac{dx}{dt} = 8\,ft/s$

• WANTED rate: $\frac{d\alpha}{dt}$ @ $z = 200\,ft$

(c) Relate and find the value at the instant. Do not forget units in your final answer.

$$\tan\alpha = \frac{100}{x} \quad \text{or} \quad x \cdot \tan\alpha = 100$$

$$\frac{d}{dt}: \quad \frac{dx}{dt}\tan\alpha + x \cdot \sec^2\alpha \cdot \frac{d\alpha}{dt} = 0$$

x @ moment: $200^2 = 100^2 + x^2 \Rightarrow x^2 = 200^2 - 100^2 = 100\sqrt{3}$

$\sec^2\alpha$ @ moment: $\tan\alpha = \frac{100}{100\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{3}$

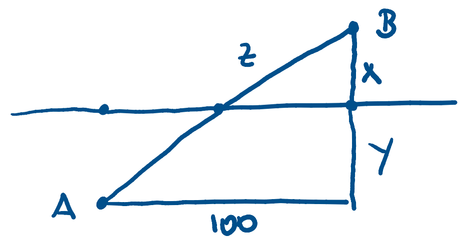
$\Rightarrow \sec^2\alpha = \frac{4}{3}$

$$\Rightarrow \frac{d\alpha}{dt} \cdot 100\sqrt{3} \cdot \frac{4}{3} = -8 \cdot \frac{1}{\sqrt{3}}$$

$$\frac{d\alpha}{dt} = \frac{-\overset{2}{\cancel{8}} \cdot \cancel{3}}{\cancel{4} \cdot 100 \cdot \cancel{3}} = -\frac{1}{50}\ rad/s$$

Problem 8 AT noon, ship A is 100km west of ship B. Ship A is sailing south at 35km/h and ship B is sailing north at 25km/h. How fast is the distance between the two ships changing at 4pm?

(a) Sketch the situation and label all relevant quantities.



(b) • KNOWN rate: $\frac{dx}{dt} = 25 \frac{km}{h}$ $\frac{dy}{dt} = 35 km/h$

• WANTED rate: $\frac{dz}{dt}$ @ 4 hours

(c) Relate and find the value at the instant. Do not forget units in your final answer.

$$100^2 + (x+y)^2 = z^2$$

$$\frac{d}{dt}: \quad 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2z\frac{dz}{dt}$$

x @ moment: $x = 4 \cdot 25 = 100$ km

y @ moment: $y = 4 \cdot 35 = 140$ km

z @ moment: $\sqrt{100^2 + (240)^2} = z = 260$ km

$$240(60) = 260 \frac{dz}{dt}$$

$$\frac{720}{13} km/h = \frac{dz}{dt}$$