MATH 124 Midterm 2

November 15, 2022

Instructor: Gaku Liu

Name: Solutions

Student #: ____________________

Problem:	1	2	3	4	5	Total
Points:	12	12	8	14	14	60

INSTRUCTIONS:

- You have 80 minutes to take the test.
- There are 5 problems. Make sure you have all of them.
- Write your solution below the problem. There is scratch paper at the back of the test.
- The test is double-sided. Make sure you are reading the backs of pages!
- Unless otherwise stated, **show all your work for full credit**.
- Unless otherwise stated, all answers should be exact, without rounding.
- You are allowed to use one 8.5"×11" sheet of notes, front and back.
- You can use a TI-30X IIS calculator. No other calculator is allowed.

TIPS:

- The number of points a question is worth is not correlated to its difficulty.
- Don't spend too much time on one problem if you haven't looked at the rest of the test.
- There is partial credit. Even if you can't fully solve a problem, explaining your progress might get you a significant number of points.
- Make sure your calculator is in radians!!!

Good luck!

1. (12 points) Find the equation of the tangent line to the curve

$$x^3 - 3xy + y^3 = 3$$

at the point $(1, 2)$.

$$\frac{d}{dx}(x^3 - 3xy + y^3) = \frac{d}{dx} 3$$

$$3x^2 - 3y - 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(-3x + 3y^2)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2-1}{4-1}$$

$$= \frac{1}{3}$$

$$\boxed{y - 2 = \frac{1}{3}(x-1)}$$

2. Let $f(x) = \ln(x^2 + x - 1)$.

(a) (6 points) Find $f'(x)$.

$$f'(x) = \frac{1}{x^2+x-1} \frac{d}{dx}(x^2+x-1)$$

$$= \boxed{\frac{2x+1}{x^2+x-1}}$$

(b) (6 points) Find the linearization of f at $a = 1$, and use it to approximate $f(1.01)$.

$$L(x) = f(1) + f'(1)(x-1)$$

$$= 0 + 3(x-1)$$

$$f(1) = \ln(1) = 0$$

$$f'(1) = \frac{3}{1} = 3$$

$$f(x) \approx 3(x-1)$$

$$f(1.01) \approx 3(0.01)$$

$$\approx \boxed{0.03}$$

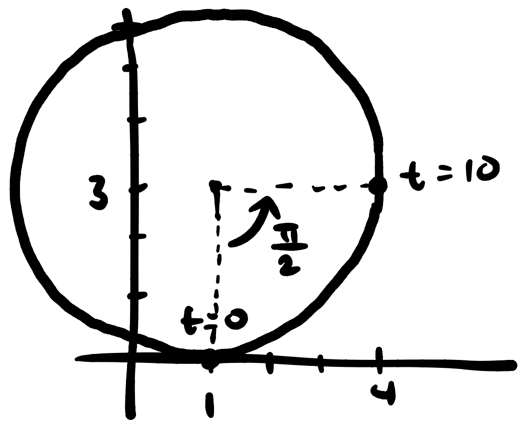
3. (8 points) A particle is moving counterclockwise along the circle $(x-1)^2+(y-3)^2=9$ at a constant speed. At the starting time $t=0$, it is at the point $(1,0)$. At time $t=10$, it reaches the point $(4,3)$ for the first time. Write parametric equations describing the motion of the particle.

center $= (1,3)$

radius $= 3$

starting angle $= -\frac{\pi}{2}$

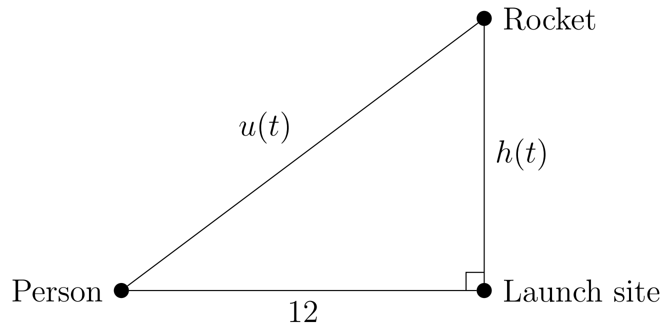
angular velocity $= \frac{\pi}{20}$



angular velocity $= \frac{\pi/2}{10} = \frac{\pi}{20}$

$$\boxed{\begin{aligned} x(t) &= 1+3\cos\left(-\frac{\pi}{2}+\frac{\pi}{20}t\right) \\ y(t) &= 3+3\sin\left(-\frac{\pi}{2}+\frac{\pi}{20}t\right) \end{aligned}}$$

4. A person is watching the launch of a toy rocket. The person is standing 12 meters away from the launch site of the rocket. At time $t = 0$ seconds, the rocket launches vertically into the air, perpendicular to the ground. Let $h(t)$ be the height of the rocket and let $u(t)$ be the distance of the rocket to the person.



(a) (10 points) When the rocket is 9 meters above the ground, it is moving upward at a speed of 8 meters per second. How fast is $u(t)$ increasing at this time? You don't need to include units in your answer.

$$u^2 = h^2 + 12^2$$

$$2u\frac{du}{dt} = 2h\frac{dh}{dt}$$

$$\frac{du}{dt} = \frac{h}{u}\frac{dh}{dt}$$

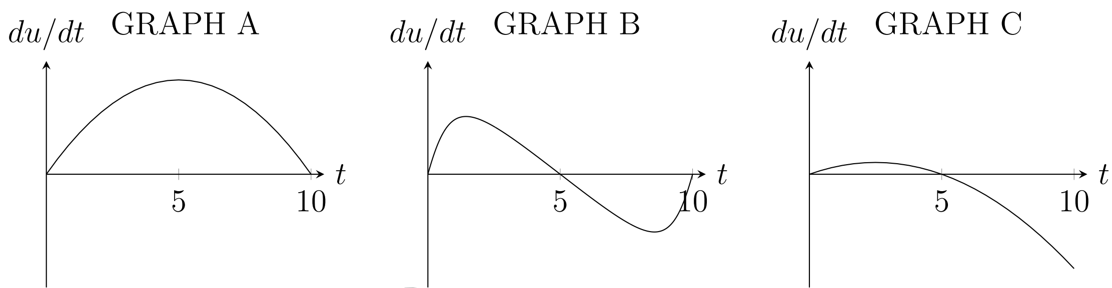
When the rocket is 9 meters above the ground,

$$u = \sqrt{9^2+12^2} = 15$$

$$\frac{du}{dt} = \frac{9}{15} \cdot 8 = \boxed{4.8}$$

PROBLEM 4 CONTINUED ON NEXT PAGE.

(b) (4 points) At $t = 5$ seconds, the rocket reaches its highest point. Afterwards, it starts falling vertically back to the ground. It hits the ground at $t = 10$ seconds. Which of the following graphs most likely resembles the graph of du/dt in the time interval $0 < t < 10$? Explain your answer.



The answer is $\boxed{B}$

We have $$\frac{du}{dt} = \frac{h}{u}\frac{dh}{dt}$$

At $t=5$, h and u are positive and $\frac{dh}{dt} = 0$ since the height is at a maximum. So $\frac{h}{u}\frac{dh}{dt} = 0$, which eliminates A.

As $t \to 10$, we have $h \to 0$, $u \to 12$, and $\frac{dh}{dt}$ approaches some negative number. Thus $\frac{h}{u}\frac{dh}{dt} \to 0$, which matches B.

Note: It is incorrect to say that because the rocket is at rest at $t=10$, its velocity should approach 0 as $t \to 10$. As $t \to 10$, the rocket is actually speeding up (or is at terminal velocity). The intuitive reason why $\frac{du}{dt} \to 0$ despite the rocket speeding up is that near the ground, its velocity is nearly perpendicular to the line from the person to the rocket, so the rocket's movement is not changing the distance to the person by much.

5. A particle is moving in the xy-plane according to the parametric equations

$$x(t) = e^{-t}\cos(5t)$$
$$y(t) = e^{-t}\sin(5t)$$

(a) (10 points) Find the slope of the tangent line to the movement of the particle when $t = \pi$.

$$\frac{dx}{dt} = -e^{-t}\cos(5t) - 5e^{-t}\sin(5t)$$

$$\frac{dy}{dt} = -e^{-t}\sin(5t) + 5e^{-t}\cos(5t)$$

$$\frac{dy}{dx} = \frac{-e^{-t}\sin(5t) + 5e^{-t}\cos(5t)}{-e^{-t}\cos(5t) - 5e^{-t}\sin(5t)}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{-e^{-\pi}(0) + 5e^{-\pi}(-1)}{-e^{-\pi}(-1) - 5e^{-\pi}(0)} = \frac{-5e^{-\pi}}{e^{-\pi}}$$

$$= \boxed{-5}$$

(b) (2 points) Let $u(t)$ be the distance of the particle from the point $(0, 0)$ at time t. Show that $u(t) = e^{-t}$.

By distance formula,

$$u(t) = \sqrt{(x(t)-0)^2 + (y(t)-0)^2}$$

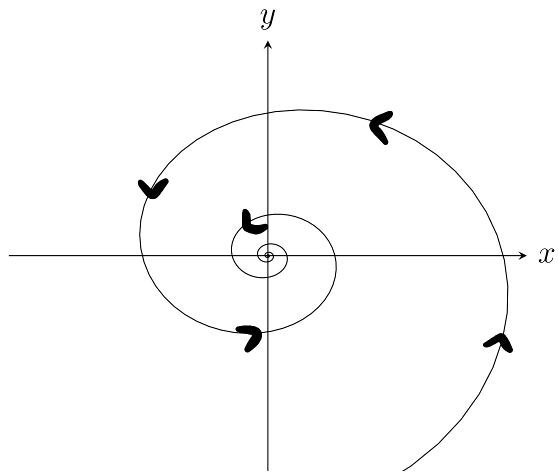
$$= \sqrt{e^{-2t}\cos^2(5t) + e^{-2t}\sin^2(5t)}$$

$$= \sqrt{e^{-2t}(\cos^2(5t) + \sin^2(5t))}$$

$$= \sqrt{e^{-2t}\cdot 1} = e^{-t}$$

PROBLEM 5 CONTINUED ON NEXT PAGE.

(c) (2 points) The following picture shows the curve traced out by the particle. Draw arrows on the curve to indicate the direction the particle moves as t increases. Explain how you got your answer.



As t increases, $u(t) = e^{-t}$ gets smaller and smaller. Thus, the particle is getting closer and closer to the origin, so its path is spiraling inward.