Math 124A, Winter 2023

Midterm II (Collingwood)

February 21, 2023

Name SLN

Student Number

Instructions.

- **These exams will be scanned. Please write your name and student number clearly for easy recognition.**
- There are 6 questions. The exam is out of 50 points.
- You are allowed to use one page of handwritten notes, 8.5 x 11, both sides ok.
- You can only use a Ti-30x IIS calculator. Unless otherwise stated, you have to give exact answers to questions. ($\frac{2\ln 3}{\pi}$ and $1/3$ are exact, 0.699 and 0.333 are approximations for the those numbers.)
- Each problem clearly states if you must show work. In cases where work is requested, **you may not get full credit for a right answer if your answer is not justified by your work.**

Question	points	Score
1	15	
2	4	
3	4	
4	10	
5	11	
6	6	
Total	50	

3pts each

1. (15 points) Find the derivatives of the following functions. Do not simplify. Your final answers must give the derivative in terms of x. Use any rules you wish. Place your final answer in the box. No work required.

(a) $y = x^\pi + \pi^x + \pi^\pi$, $\frac{dy}{dx} = \boxed{\underbrace{\pi x^{\pi-1}}_{1} + \underbrace{\ln\pi \cdot \pi^x}_{1} + \underbrace{0}_{1}}$

(b) $y = (2x^3 - 4x + 1)^{35}$, $\frac{dy}{dx} = \boxed{\underbrace{35(2x^3-4x+1)^{34}}_{2} \cdot \underbrace{(6x^2-4)}_{1}}$

(c) $y = \sin^2(\sqrt{x})$, $\frac{dy}{dx} = \boxed{\underbrace{2\sin(\sqrt{x})}_{1} \cdot \underbrace{\cos(\sqrt{x})}_{1} \cdot \underbrace{\left(\frac{1}{2\sqrt{x}}\right)}_{1}}$

(d) $y = \sqrt{\sin(\pi x)}$, $\frac{dy}{dx} = \boxed{\underbrace{\frac{1}{2\sqrt{\sin \pi x}}}_{1} \cdot (\underbrace{\pi}_{1} \underbrace{\cos \pi x}_{1})}$

(e) $y = \frac{e^{\sqrt{x}}}{x}$, $\frac{dy}{dx} = \boxed{\frac{x \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} - e^{\sqrt{x}}}{x^2}}$ 2 1pt

2. (4 points) Compute the slope of the tangent line to the curve $y = x^{\sin(\pi x)}$ at the point where $x = 3/2$. Give EXACT answers.

① $$y = x^{\sin \pi x}$$

$$\ln y = \ln x^{\sin \pi x} = \sin \pi x \cdot \ln x$$

differentiate

② $$\frac{1}{y} y' = (\sin \pi x)' \ln x + \sin \pi x (\ln x)'$$

$$\frac{y'}{y} = \pi \cos \pi x \cdot \ln x + \frac{\sin \pi x}{x}$$

$$y' = x^{\sin \pi x} \left(\pi \cos \pi x \cdot \ln x + \frac{\sin \pi x}{x} \right)$$

① $$y'\Big|_{3/2 = x} = (3/2)^{\sin \frac{3\pi}{2}} \left(\pi \cdot \cos \frac{3\pi}{2} \cdot \ln 3/2 + \frac{\sin \frac{3\pi}{2}}{\frac{3}{2}} \right)$$

$$= (3/2)^{-1} \left(0 + \frac{-1}{3/2} \right) =$$

$$= \left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{-4}{9}$$

3. (4 points) Suppose

$$F(x) = \arctan\left(\frac{10}{x}\right) - \arctan\left(\frac{2}{x}\right)$$

Find ALL x so that $F'(x) = 0$. (Recall, $\arctan(\ldots) = tan^{-1}(\ldots)$.)

2

$$F'(x) = \frac{1}{1+(\frac{10}{x})^2} \cdot \left(\frac{-10}{x^2}\right) - \frac{1}{1+(\frac{2}{x})^2}\left(\frac{-2}{x^2}\right)$$

$$F'(x) = \frac{-10}{x^2+100} + \frac{2}{x^2+4}$$

1

$$0 = \frac{-10}{x^2+100} + \frac{2}{x^2+4}$$

$$\frac{10}{x^2+100} = \frac{2}{x^2+4}$$

$$10x^2 + 40 = 2x^2 + 200$$

$$8x^2 = 160$$

$$x^2 = 20$$

$$x = \pm\sqrt{20} = \pm 2\sqrt{5}$$

1

4. (10 points) An object is moving in the plane along a curve with parametric equations

$$x(t) = \frac{2}{t^2+1}, \qquad y(t) = \frac{3t-1}{2t+1},$$

where $t \geq 0$ is in units of seconds and the units on the coordinate axes are feet. Put your answers in the boxes. Some work required for each part to receive credit. Give EXACT answers or three decimal places of accuracy.

(2) (a) (True or False) The vertical velocity at time t is always positive. Explain.

$$y'(t) = \frac{(2t+1)\cdot 3 - (3t-1)\cdot 2}{(2t+1)^2} = \frac{5}{(2t+1)^2}$$ ← positive for all t ← positive for all t

$\Rightarrow y'(t) > 0$

(1) pt deriv. calc.

(b) At time t, what is the slope of the curve at the object location?

(3) $$\boxed{\frac{-5(t^2+1)^2}{4t(2t+1)^2}}$$

$$s(t) = \text{slope @ } p(t) = \frac{y'(t)}{x'(t)} = \frac{\frac{5}{(2t+1)^2}}{\frac{-2\cdot 2t}{(t^2+1)^2}} = \frac{-5(t^2+1)^2}{4t(2t+1)^2}$$

(c) What is the initial speed of the object (i.e. speed at time $t = 0$)?

(1) $\boxed{5}$

$$\text{speed}(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{\left(\frac{-4t}{(t^2+1)^2}\right)^2 + \left(\frac{5}{(2t+1)^2}\right)^2}$$

$$\text{speed}(0) = \sqrt{0^2 + \frac{5^2}{1}} = 5$$

(d) What is the equation of the tangent line to the path when the object crosses the x-axis?

(4)

(1) $y(t) = 0 \Leftrightarrow 3t - 1 = 0 \Rightarrow t = 1/3$

(1) (*) $\text{position}(\tfrac{1}{3}) = P(1/3) = (x(1/3), y(1/3)) = \left(\frac{2}{\frac{10}{9}}, 0\right) = (9/5, 0)$

(1) (*) $\text{slope}(1/3) = \frac{-5(\frac{1}{9}+1)^2}{4/3\,(5/3)^2} = \frac{-5\cdot\frac{100}{81}\cdot 27}{4\cdot 25} = -5/3$

(1) TL: $y = -5/3(x - 9/5) + 0$

5. (11 points) The equation

$$x^2 - 4xy + 8y^2 = 1$$

defines an ellipse. You must show work on each part to receive credit. Give EXACT answers.

(a) Find a formula (in terms of x and y) for the implicit derivative $\frac{dy}{dx}$.

3

$$2x - 4y - 4xy' + 16yy' = 0$$

$$(16y - 4x)\,y' = 4y - 2x$$

$$y' = \frac{dy}{dx} = \frac{4y-2x}{16y-4x} = \frac{2y-x}{8y-2x}$$

either ok.

one point

(b) Find ~~ALL~~ points on the ellipse where the tangent line is parallel to the line $x + y = 1$.

4

1 $\Rightarrow y = -x + 1 \Rightarrow$ slope $= -1$.

1 $$-1 = \frac{2y-x}{8y-2x} \Rightarrow 2x - 8y = 2y - x \Rightarrow 3x = 10y \Rightarrow \boxed{\tfrac{3}{10}x = y}$$

1 Plug into orig eqn: $x^2 - 4x\left(\tfrac{3}{10}x\right) + 8\left(\tfrac{3}{10}x\right)^2 = 1$

$$\left(1 - \tfrac{6}{5} + \tfrac{18}{25}\right)x^2 = 1 \Rightarrow x^2 = \tfrac{25}{13}$$

1 $$\Rightarrow x = \pm\frac{5}{\sqrt{13}} \Rightarrow Pt_1 = \left(\frac{5}{\sqrt{13}}, \frac{15}{10\sqrt{13}}\right),\ Pt_2 = \left(\frac{-5}{\sqrt{13}}, \frac{-15}{10\sqrt{13}}\right)$$

Either point ok.

(c) What is the slope of the tangent line to the curve at the point $P = (1, 1/2)$?

1 $$y'|_P = \frac{2\cdot\frac{1}{2} - 1}{8\cdot\frac{1}{2} - 2} = 0$$

(d) What is the value of the second implicit derivative at the point $P = (1, 1/2)$; i.e. $y''|_P = \frac{d^2y}{dx^2}|_P$?

3

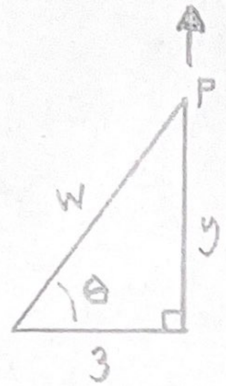
$$y'' = \frac{(8y-2x)(2y-x)' - (2y-x)(8y-2x)'}{(8y-2x)^2}$$

$$y'' = \frac{(8y-2x)(2y'-1) - (2y-x)(8y'-2)}{(8y-2x)^2}$$

$$y''|_P = \frac{(8\cdot\frac{1}{2}-2)(2\cdot 0 - 1) - (2\cdot\frac{1}{2}-1)(8\cdot 0 - 2)}{(8\cdot\frac{1}{2}-2)^2}$$

$$= \frac{-2 - 0}{4} = -\frac{1}{2}$$

6. (6 points) The pictured right triangle has sides y and w which are elastic and will stretch when the vertex P is pulled in the direction indicated. Assume that as P is pulled in the picture, the triangle is always a right triangle with base 3 feet. Show some work. Give EXACT answers or answers accurate to 3 decimal places.



(a) Suppose y is increasing at a rate of 7 ft/sec. When $y = 5$ feet, what is the rate at which w is changing?

3

1

$w = \sqrt{9+y^2}$

$$\frac{dw}{dt} = \underbrace{\frac{2y\frac{dy}{dt}}{2\sqrt{9+y^2}}}_{1} \Rightarrow \underbrace{\left.\frac{dw}{dt}\right|_{\substack{y=5\\ y'=7}} = \frac{2\cdot5\cdot7}{2\sqrt{9+25}} = \frac{35}{\sqrt{34}} \text{ ft/s.}}_{1}$$

(b) Suppose θ is increasing at a rate of 0.1 rad/sec. When $\theta = \frac{\pi}{3}$ radians, what is the rate at which w is changing?

3 pts

$\cos\theta = \frac{3}{W}$

$-\sin\theta \frac{d\theta}{dt} = \frac{-3\frac{dw}{dt}}{w^2}$

$\frac{dw}{dt} = \frac{-\sin\theta \cdot w^2 \cdot \frac{d\theta}{dt}}{-3}$

$\left.\frac{dw}{dt}\right|_{\substack{\theta=\frac{\pi}{3}\\ d\theta/dt=0.1}} = \frac{\sin\frac{\pi}{3}\cdot 6^2 \cdot \frac{1}{10}}{3}$

$= \frac{\sqrt{3}}{2}\cdot\frac{12}{10} = \frac{12\sqrt{3}}{20} = \frac{3\sqrt{3}}{5}$

$\cos\frac{\pi}{3} = \frac{3}{W}$

$W = \frac{3}{\cos\frac{\pi}{3}} = \frac{3}{\frac{1}{2}}$

$= 6$