Math 124 B C Midterm 2

Prof. Charles Camacho **Math 124 B & C** **Midterm 2 Exam, Autumn 2023**

Print Your Full Name	Signature
Solutions	
Student ID Number	Quiz Section
Instructor's Name	TA's Name

Please read these instructions!

1. Your midterm exam contains 7 pages; PLEASE MAKE SURE YOU HAVE A COMPLETE EXAM.
2. You are allowed a single, double-sided 8.5"x11" handwritten notesheet; the TI-30XIIS calculator; a writing utensil; and an eraser to use on the exam.
3. The entire midterm exam is worth 80 points. Point values for problems vary and these are clearly indicated. You have 80 minutes for this exam.
4. Make sure to ALWAYS SHOW YOUR WORK CLEARLY. Credit is awarded to work which is clearly shown and legible. Partial credit is awarded as earned. Full credit may not be awarded if work is unclear or illegible.
5. For problems that aren't sketches, place a box around your final answer to each question. Do not box in multiple, different final answers, as this may result in loss of credit.
6. There is plenty of space on the exam to do your work. If you need extra space, use the last two pages of the exam. Clearly indicate that there is more work located on the last pages, and indicate on those pages the related problem number.
7. Unless otherwise instructed, always give your answers in exact form. For example, 3π, $\sqrt{2}$, and $\ln(2)$ are in exact form; the corresponding approximations 9.424778, 1.4142, and 0.693147 are NOT in exact form.
8. Credit is awarded for correct use of techniques or methods discussed in class thus far. Partial credit may be awarded as earned. No credit is awarded for use of methods that are learned later in the course.

Problem	Total Points	Score
1	20	
2	20	
3	20	
4	20	
Total	80	

1. Find the derivatives of the following functions. Do not simplify your answers.

(a) (5 points) $y = \dfrac{x}{\sqrt{x^2 - 5x}}$

$$y' = \frac{\sqrt{x^2-5x}\cdot 1 - x\cdot \frac{1}{2\sqrt{x^2-5x}}(2x-5)}{x^2-5x}$$

(b) (5 points) $y = \sin^{-1}(x^{3/2})$

$$y' = \frac{1}{\sqrt{1-(x^{3/2})^2}}\cdot \frac{3}{2}x^{\frac{1}{2}}$$

(c) (5 points) $y = \sin^4(\tan(x)) = (\sin(\tan x))^4$

$$y' = 4(\sin(\tan x))^3\cdot \cos(\tan x)\cdot \sec^2 x$$

(d) (5 points) $y = \sqrt{x}^{\sqrt{x}}$

$$\ln y = \ln\left(\sqrt{x}^{\sqrt{x}}\right)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\sqrt{x}\cdot \ln(\sqrt{x})\right)$$

$$\frac{1}{y}\cdot y' = \sqrt{x}\cdot \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} + \ln(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

$$y' = \sqrt{x}^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}} + \frac{\ln(\sqrt{x})}{2\sqrt{x}}\right)$$

2. (20 points) Find an equation of the tangent line to the curve

$$2x^2y + xy^2 = \ln(x^2)$$

at the point with x-coordinate $x = 1$ and negative y-coordinate.

$$\frac{d}{dx}\left(2x^2y + xy^2\right) = \frac{d}{dx}\left(\ln(x^2)\right)$$

$$2x^2 \cdot \frac{dy}{dx} + y \cdot 4x + x \cdot 2y\frac{dy}{dx} + y^2 = \frac{1}{x^2} \cdot 2x$$

$$\frac{dy}{dx}\left(2x^2 + 2xy\right) = \frac{2}{x} - 4xy - y^2$$

$$\frac{dy}{dx} = \frac{\frac{2}{x} - 4xy - y^2}{2x^2 + 2xy}$$

$x = 1 \Rightarrow 2(1)^2y + y^2 = \ln(1)$

$2y + y^2 = 0$ (since $y < 0$)

$y(2 + y) = 0 \rightarrow y = -2$

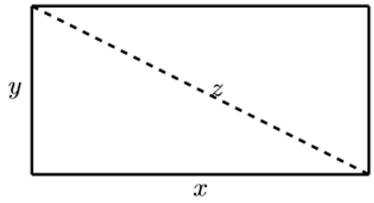
$$\Rightarrow \frac{dy}{dx} = \frac{\frac{2}{1} - 4(1)(-2) - (-2)^2}{2(1)^2 + 2(1)(-2)} = \frac{2 + 8 - 4}{2 - 4}$$

$$= \frac{6}{-2} = -3$$

$\Rightarrow$ Eq. of tangent line is

$$\boxed{y = -3(x - 1) - 2}$$

3. (20 points) A rectangular sheet of rubber with area 120 cm^2 is to be constructed with a decorative ribbon along a diagonal (see the figure below). Suppose you can stretch a corner of the rubber so that the area of the sheet stays the same while one side, x, increases at the constant rate of 5 cm/sec. Find the rate of change of the ribbon's length z when the stretched side x reaches 40 cm.



Given

- $A = 120$
- $\frac{dx}{dt} = 5 \frac{cm}{s}$

Wanted

$\frac{dz}{dt}$ when $x = 40$ cm

Equation: $x^2 + y^2 = z^2$ $\quad A = xy = 120$

$y = \frac{120}{x}$

$\Rightarrow x^2 + \frac{120^2}{x^2} = z^2$

$\frac{d}{dt}\left(x^2 + \frac{120^2}{x^2}\right) = \frac{d}{dt}(z^2)$

$2x \cdot \frac{dx}{dt} + 120^2 (-2) x^{-3} \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$

$\frac{dx}{dt}\left(\frac{x - 120^2 \cdot x^{-3}}{z}\right) = \frac{dz}{dt}$

$x = 40 \rightarrow y = \frac{120}{40} = 3 \rightarrow 40^2 + 3^2 = z^2 \rightarrow z = \sqrt{1609}$

$\Rightarrow 5\left(\frac{40 - 120^2 (40)^{-3}}{\sqrt{1609}}\right) \frac{cm}{s} = \frac{dz}{dt}$

$\frac{dz}{dt} \approx \boxed{4.96 \frac{cm}{s}}$

4. (20 points) Consider the curve in the xy-plane with parametric equations

$$x = t^3 - t, \quad y = -t^2 - 3, \quad -\infty < t < \infty.$$

Find all the (x, y) points on the curve whose tangent lines have slope equal to 1.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2t}{3t^2 - 1} = 1$$

$$-2t = 3t^2 - 1$$

$$0 = 3t^2 + 2t - 1$$

$$= (3t - 1)(t + 1)$$

$$t = \frac{1}{3}, -1$$

$t = \frac{1}{3}$: $x = \left(\frac{1}{3}\right)^3 - \frac{1}{3} = \frac{1}{3}\left(\frac{1}{9} - 1\right) = \frac{-8}{27}$ $\quad y = -\left(\frac{1}{3}\right)^2 - 3 = -\frac{1}{9} - 3 = \frac{-28}{9}$

$t = -1$: $x = (-1)^3 - (-1) = -1 + 1 = 0$ $\quad y = -(-1)^2 - 3 = -1 - 3 = -4$

$$\Rightarrow \boxed{\left(\frac{-8}{27}, \frac{-28}{9}\right) \text{ and } (0, -4)}$$

Extra scratch paper.

Extra scratch paper.