Solutions to Math 124 G Fall 2025 Midterm II

1. (a) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{9x+\sqrt{8x+7e^x}}}\left(9+\dfrac{8+7e^x}{2\sqrt{8x+7e^x}}\right)$

   (b) $\dfrac{dy}{dx} = e^{\tan x}\cdot\sec^2 x + e^x\sec^2(e^x) + e(\tan x)^{e-1}\sec^2 x$

   (c) $\ln y = (3x+4)\cdot\ln(1+2x^2)$

$$\frac{y'}{y} = 3\ln(1+2x^2) + \frac{2x(3x+4)}{1+2x^2}$$

$$y' = \left(3\ln(1+2x^2) + \frac{2x(3x+4)}{1+2x^2}\right)\cdot(1+2x^2)^{(3x+4)}$$

2. Note that

$$2y^2x^3 + \ln\left(\frac{2x}{y}\right) = 2y^2x^3 + \ln 2 + \ln x - \ln y$$

   (a) Differentiate:

$$4yy'x^3 + 2y^2\cdot 3x^2 + \frac{1}{x} - \frac{y'}{y} = 0$$

   when $x=1$ and $y=2$

$$8y' + 24 + 1 - \frac{y'}{2} = 0$$

   So $y' = -\frac{10}{3}$ so the tangent line is

$$y - 2 = -\frac{10}{3}(x-1)$$

   and when $x = 1.05$

$$y \approx 2 - \frac{10}{3}(1.05-1) = \frac{11}{6}$$

   (b) Differentiate again:

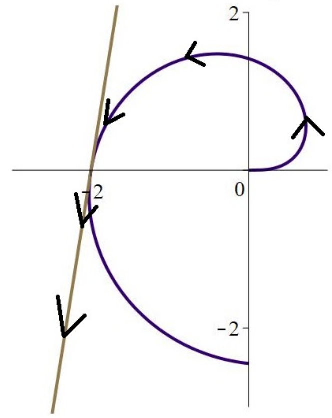
$$4y'y'x^3 + 4yy''x^3 + 4yy'3x^2 + 4yy'\cdot 3x^2 + 2y^2\cdot 6x - \frac{1}{x^2} - \frac{y''y - y'y'}{y^2} = 0$$

   when $x=1$, $y=2$, and $y' = -10/3$

$$\frac{400}{9} + 8y'' - 80 - 80 + 48 - 1 - \frac{2y'' - \frac{100}{9}}{4} = 0$$

   we get $\frac{15}{2}y'' = 113 - \frac{425}{9} > 0$ so the graph is concave up and $\frac{11}{6}$ is less than the actual value of y.

3. (a)



$$x'(t) = \frac{dx}{dt} = \frac{1}{2\sqrt{t}}\cos\left(\frac{\pi}{4}t\right) - \frac{\pi}{4}\sqrt{t}\sin\left(\frac{\pi}{4}t\right)$$

so

$$x'(2) = \frac{1}{2\sqrt{2}}\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{4}\sqrt{2}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi\sqrt{2}}{4}$$

and

$$y'(t) = \frac{dy}{dt} = \frac{1}{2\sqrt{t}}\sin\left(\frac{\pi}{4}t\right) + \frac{\pi}{4}\sqrt{t}\cos\left(\frac{\pi}{4}t\right)$$

so

$$y'(2) = \frac{1}{2\sqrt{2}}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}\sqrt{2}\cos\left(\frac{\pi}{2}\right) = \frac{1}{2\sqrt{2}}$$

therefore, the speed is

$$\sqrt{(x'(2))^2 + (y'(2))^2} = \sqrt{\left(-\frac{\pi\sqrt{2}}{4}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} = \sqrt{\frac{\pi^2+1}{8}}$$

(b) The distance to the origin is

$$z = \sqrt{(x(t))^2 + (y(t))^2} = \sqrt{t\cos^2\left(\frac{\pi}{4}t\right) + t\sin^2\left(\frac{\pi}{4}t\right)} = \sqrt{t}$$

so

$$\frac{dz}{dt} = \frac{1}{2\sqrt{t}}$$

and at $t = 3$ we get $\frac{1}{2\sqrt{3}}$.

(c) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{t}}\sin\left(\frac{\pi}{4}t\right) + \frac{\pi}{4}\sqrt{t}\cos\left(\frac{\pi}{4}t\right)}{\frac{1}{2\sqrt{t}}\cos\left(\frac{\pi}{4}t\right) - \frac{\pi}{4}\sqrt{t}\sin\left(\frac{\pi}{4}t\right)}$$

When $t = 4$ we get

$$\left.\frac{dy}{dx}\right|_{t=4} = \frac{\frac{1}{2\sqrt{4}}\sin(\pi) + \frac{\pi}{4}\sqrt{4}\cos(\pi)}{\frac{1}{2\sqrt{4}}\cos(\pi) - \frac{\pi}{4}\sqrt{4}\sin(\pi)} = \frac{-\frac{\pi}{2}}{-\frac{1}{4}} = 2\pi$$

The point of tangency is given by

$$x(4) = \sqrt{4}\cos(\pi) = -2 \qquad y(4) = \sqrt{4}\sin(\pi) = 0$$

so the equation of the tangent line is

$$y - 0 = 2\pi(x+2)$$

4. For the cone, let y be the depth of the coffee and r be the radius at the surface of the coffee then the volume of coffee in the cone is

$$V = \frac{1}{3}\pi r^2 y$$

and using similar triangles

$$\frac{5}{r} = \frac{12}{y}$$

so

$$V = \frac{1}{3}\pi \left(\frac{5y}{12}\right)^2 y = \frac{25\pi}{3 \cdot 12^2} y^3$$

then,

$$\frac{dV}{dt} = \frac{25\pi}{3 \cdot 12^2} 3y^2 \frac{dy}{dt} = \frac{25\pi}{12^2} y^2 \frac{dy}{dt}$$

When $y = 6$ and $dy/dt = 1.5$ the rate at which coffee leaves the cone is

$$\frac{dV}{dt} = \frac{25\pi}{12^2} 6^2 \cdot 1.5 = \frac{75\pi}{8} \text{ cubic centimeters per minute}$$

which is the same rate it enters the cylindrical mug. Let h be the height of the coffee in the mug. The radius of the mug is 3 so the volume of coffee in the mug is

$$V = \pi \cdot 3^3 \cdot h$$

so

$$\frac{75\pi}{8} = \frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

so $\frac{dh}{dt} = \frac{25}{24}$ centimeters per minute.