Math 124D – Midterm II

Autumn 2021

Solutions

Name: ____________________

Student ID #: ____________________

Question	Points	Score
1	12	
2	12	
3	10	
4	12	
5	4	
Total:	50	

1. (12 points) Find the derivatives $y'(x) = \frac{dy}{dx}$ of the following functions.
(Your final answers should be in terms of x only.)

(a) $y = \sqrt{x \sin(5x)}$

Solution: We can write $\sqrt{x\sin(5x)} = (x\sin(5x))^{1/2}$. Then

$$\tfrac{d}{dx}\sqrt{x\sin(5x)} = \frac{1}{2}(x\sin(5x))^{-1/2} \cdot \tfrac{d}{dx}\,(x\sin(5x))$$
$$= \frac{1}{2}(x\sin(5x))^{-1/2} \cdot (\sin(5x) + x\cos(5x)\cdot 5) = \frac{\sin(5x) + 5x\cos(5x)}{2\sqrt{x\sin(5x)}}$$

(b) $y = \arctan\left(\frac{x^3}{x+1}\right)$ (Here $\arctan(x)$ denotes the inverse function of $\tan(x)$.)

Solution:

$$\frac{d}{dx}\left[\arctan\left(\frac{x^3}{x+1}\right)\right] = \frac{1}{\left(\frac{x^3}{x+1}\right)^2 + 1} \cdot \frac{(x+1)(3x^2) - x^3(1)}{(x+1)}^2 = \frac{2x^3 + 3x^2}{x^6 + (x+1)^2}.$$

(c) $y = x^{(e^x)}$

Solution: First we apply $\ln(\cdot)$ to both sides,

$$\ln(y) = \ln(x^{e^x}) = e^x \ln(x).$$

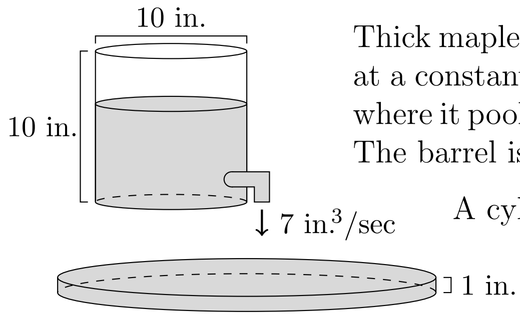
Differentiating both sides with respect to x the gives

$$\frac{1}{y}\frac{dy}{dx} = e^x \ln(x) + e^x \cdot \frac{1}{x} = e^x\left(\ln(x) + \frac{1}{x}\right)$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = y \cdot e^x\left(\ln(x) + \frac{1}{x}\right) = x^{(e^x)} e^x\left(\ln(x) + \frac{1}{x}\right).$$

2. (12 points)



Thick maple syrup starts oozing out of a full cylindrical barrel at a constant rate of 7 cubic inches per second onto the floor, where it pools in cylindrical shape of uniform 1-inch thickness. The barrel is 10 inches in both height and diameter.

A cylinder of radius r and height h has volume $\pi r^2 h$.

Include units in your answers!

(a) What is the rate of change of the height of syrup in the barrel when the syrup level is 4 inches from the top of the barrel?

Solution: Let V denote the volume of syrup in the barrel and let h denote its height. The radius of the syrup is the same as the barrel, 5 inches. Then

$$V = \pi r^2 h = \pi \cdot (5)^2 h = 25\pi h.$$

Differentiating both sides with respect to t, we find that $\frac{dV}{dt} = 25\pi \cdot \frac{dh}{dt}$. Since $\frac{dV}{dt}$ equals -7 in^3/sec, we can solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-7}{25\pi} \text{ inches per second .}$$

Alternatively, the height is decreasing at a rate of $\frac{7}{25\pi}$ inches per second.

(b) At what rate is the diameter of the maple syrup on the floor changing at this time?

Solution: Let V denote the volume of syrup on the floor and r its radius. The $V = \pi r^2(1) = \pi r^2$ Differentiating with respect to t gives $\frac{dV}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$. We know that $\frac{dV}{dt} = 7$.

When the syrup level is 4 inches from the top, it has lost $25\pi \cdot 4 = 100\pi$ cubic inches of maple syrup, which is now on the floor. We can solve for the radius of the maple syrup:

$$100\pi = \pi r^2 \quad \Rightarrow \quad r = \sqrt{100} = 10.$$

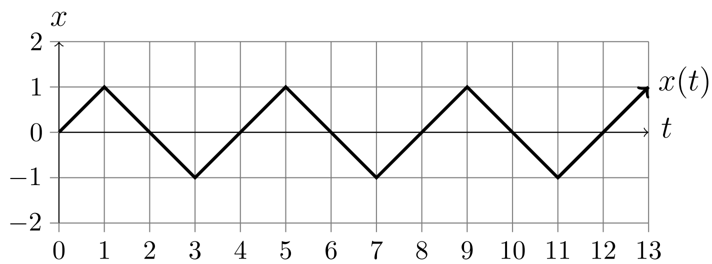
Therefore

$$7 = \pi \cdot 2(10)\frac{dr}{dt} = 20\pi \frac{dr}{dt}.$$

Solving for $\frac{dr}{dt}$ gives $\frac{dr}{dt} = \frac{7}{20\pi}$. If ℓ is the diameter of the syrup, then $\ell = 2r$ so $\frac{d\ell}{dt} = 2\frac{dr}{dt}$. Then

$$\frac{d\ell}{dt} = 2 \cdot \frac{dr}{dt} = \frac{7}{10\pi} \quad \text{inches per second.}$$

3. (10 points) Consider the curve parametrized by $(x(t), y(t))$ where $x(t)$ is graphed below and $y(t) = \cos(\pi t/2)$.



(a) Find a formula for $y'(x) = \dfrac{dy}{dx}$ when $1 < t < 3$.

Solution: (a) We compute that $\frac{dy}{dt} = -\sin(\pi t/2)(\frac{\pi}{2}) = -\frac{\pi}{2}\sin(\pi t/2)$. From the graph of $x(t)$ we see that for $1 < t < 3$, $\frac{dx}{dt} = -1$. Therefore for $1 < t < 3$,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-\frac{\pi}{2}\sin(\pi t/2)}{(-1)} = \frac{\pi}{2}\sin(\pi t/2).$$

(b) Find $y''(x) = \dfrac{d^2y}{dx^2}$ when $t = 2$.

Solution: (b) For $1 < t < 3$, the derivative $\frac{dy}{dx}$ equals $\frac{\pi}{2}\sin(\pi t/2)$, so

$$y''(x) = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{\pi}{2}\cos(\pi t/2)\cdot(\frac{\pi}{2})}{(-1)} = \frac{-\pi^2}{4}\cos(\pi t/2).$$

At $t = 2$, this gives $(-\pi^2/4)\cos(\pi) = (-\pi^2/4)\cos(\pi) = \pi^2/4$.

(c) What is the smallest positive value of t for which the point $(x(t), y(t))$ returns to its starting position at $t = 0$?

Solution: (c) For the point to come back to its starting position, we must have $x(t) = 0$, which occurs for $t = 2, 4, 6, 8, 10, 12$.
At $t = 2$, $y(t) = \cos(\pi) = -1 \neq 1 = y(0)$, so the point has not returned to its starting position.
At $t = 4$, $y(t) = \cos(2\pi) = 1 = y(0)$. Therefore the first time is $t = 4$.

4. (12 points) Consider the implicit equation $y^7 - y = x$.

(a) Find a formula for $y'(x) = \dfrac{dy}{dx}$ in terms of x and y.

Solution: Differentiating both sides with respect to x, we find that

$$7y^6 \cdot y'(x) - y'(x) = 1 \quad \Rightarrow \quad y'(x)(7y^6 - 1) = 1 \quad \Rightarrow \quad y'(x) = \frac{1}{7y^6 - 1} = (7y^6 - 1)^{-1}$$

(b) Find a formula for $y''(x) = \dfrac{d^2y}{dx^2}$ in terms of x and y.

Solution: Differentiating with respect to x again then gives

$$\begin{aligned} y''(x) = \frac{d}{dx}\left[(7y^6 - 1)^{-1}\right] &= -(7y^6 - 1)^{-2} \cdot (42 \cdot y^5) \cdot y'(x) \\ &= -(7y^6 - 1)^{-2} \cdot (42 \cdot y^5) \cdot (7y^6 - 1)^{-1} \\ &= \frac{-42y^5}{(7y^6 - 1)^3}. \end{aligned}$$

(c) Use a linear approximation of the curve at $(0, 1)$ to estimate a value of y so that the point $(\frac{1}{2}, y)$ lies on the curve.

Solution: At $(x, y) = (0, 1)$,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = \frac{1}{7(1)^6 - 1} = \frac{1}{6}.$$

The linear approximation is then $L(x) = 1 + \frac{1}{6}x$ and the approximate y-value is

$$L(1/2) = 1 + \frac{(1/2)}{6} = 1 + \frac{1}{12} = \frac{13}{12}.$$

(d) Does your answer from part (c) over- or under-estimate the true value?

Solution: Note that when $y \geq 1$, the second derivative $y''(x)$ is negative. Therefore the answer in part (d) overestimates the true value of y.

5. (4 points) From the definition of the derivative, the following limit can be written as $f'(a)$ for some function $f(x)$ and value $x = a$:

$$\lim_{h\to 0} \ln((1+h)^{1/h}) = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h}.$$

(a) What is a choice of $f(x)$ and a so that this limit equals $f'(a)$?

Solution: We can take the function $f(x) = \ln(x)$ and value $a = 1$.

Alternatively,
$f(x) = \ln(1+x)$ and $a = 0$ or
$f(x) = \ln(1-c+x)$ and $a = c$ for any c.

(b) Use rules of differentiation to find $f'(x)$ and evaluate to find the above limit.

Solution: The derivative of $\ln(x)$ is $f'(x) = \frac{1}{x}$ and so the limit equals $f'(1) = 1$.

(c) Use $e^{\ln(x)} = x$ and your answer in part (b) to find $\lim_{h\to 0}(1+h)^{1/h}$.
(Keep your answer in exact form, not numerical.)

Solution: Since $\ln((1+h)^{1/h}) \to 1$ as $h \to 0$ and the function e^x is continuous, the limit of the function $(1+h)^{1/h} = e^{\ln((1+h)^{1/h})}$ as $h \to 0$ is $e^1 = e$.