1. (12 total points) Compute the derivatives of the following functions. Do not simplify your answers.

(a) (4 points) $f(x) = \sqrt{\cos^2 x + 5x^7}$

$$f'(x) = \frac{-2\cos(x)\sin(x) + 35x^6}{2\sqrt{\cos^2 x + 5x^7}}$$

(b) (4 points) $g(t) = \tan^{-1}\left(\frac{5t+3}{t^2+4}\right)$

$$\begin{aligned} g'(t) &= \frac{1}{1+\left(\frac{5t+3}{t^2+4}\right)^2} \cdot \frac{5\cdot(t^2+4) - 2t\cdot(5t+3)}{(t^2+4)^2} \\ &= \frac{-5t^2 - 6t + 20}{(t^2+4)^2 + (5t+3)^2} \end{aligned}$$

(c) (4 points) $h(x) = \sin\left(2 + \sin\sqrt{1+x^3}\right)$

$$h'(x) = \cos\left(2 + \sin\sqrt{1+x^3}\right) \cdot \cos\sqrt{1+x^3} \cdot \frac{3x^2}{2\sqrt{1+x^3}}$$

2. (12 total points) An object is moving along an ellipse. Its location is given by the parametric equations

$$x(t) = 1 + 2\cos t \qquad y(t) = 2 + 4\sin t$$

In this problem we take $0 \le t \le 2\pi$.

(a) (6 points) Find a formula that gives the slope of the tangent line to the path at time t as a function of t.

$$\begin{aligned} \frac{dy}{dt} &= 4\cos t \\ \frac{dx}{dt} &= -2\sin t \\ m_{\tan} = \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{4\cos t}{-2\sin t} \\ &= -2\cot t \end{aligned}$$

(b) (6 points) Find the equation of the tangent line at $t = \frac{\pi}{3}$.

$$y - b = m(x - a)$$

$$\begin{aligned} a = x(\pi/3) &= 1 + 2\cos(\pi/3) = 2 \\ b = y(\pi/3) &= 2 + 4\sin(\pi/3) = 2 + 2\sqrt{3} \\ m = \left.\frac{dy}{dx}\right|_{t=\pi/3} &= -2\cot(\pi/3) = -\frac{2}{\sqrt{3}} \end{aligned}$$

$$y - 2 - 2\sqrt{3} = -\frac{2}{\sqrt{3}}(x - 2)$$

3. (8 points) Compute the equation of the tangent line to the curve $y = x^{\cos(\pi x)}$ at the point where $x = 2$.

The tangent line is of the form $y - b = m(x - a)$

Here $a = 2$ *and* $b = 2^{\cos 2\pi} = 2$

Now we compute m:

$$\begin{aligned} y &= x^{\cos(\pi x)} \\ \ln(y) &= \ln\left(x^{\cos(\pi x)}\right) \\ &= \cos(\pi x) \cdot \ln(x) \\ \frac{y'}{y} &= -\pi \sin(\pi x) + \cos(\pi x) \cdot \frac{1}{x} \\ m &= y' \Big|_{x=2, y=2} \\ &= 2 \cdot \left(-\pi \sin(2\pi) + \cos(2\pi) \cdot \frac{1}{2}\right) \\ &= 1 \end{aligned}$$

The tangent line has equation $y - 2 = 1 \cdot (x - 2)$ *or* $y = x$

4. (8 points) If $x^3 + xy + y^2 = 8$, find the value of y'' at all points where $x = 2$.

If $x = 2$ *then* $8 + 2y + y^2 = 8$ *and we get* $y = -2, 0$ *so there are two points:* $(2, -2)$ *and* $(2, 0)$.

Differentiating both sides both respect to x gives $3x^2 + y + xy' + 2yy' = 0$

At the point $(2, -2)$, $12 - 2 + 2y' - 4y' = 0$ *so* $y' = 5$
At the point $(2, 0)$, $12 + 0 + 2y' + 0y' = 0$ *so* $y' = -6$

Differentiate both sides again to get $6x + y' + y' + xy'' + 2y'y' + 2yy'' = 0$

At the point $(2, -2)$, $12 + 2 \cdot 5 + 2y'' + 2 \cdot 5^2 - 4y'' = 0$ *so* $y'' = 36$
At the point $(2, 0)$, $12 - 2 \cdot 6 + 2y'' + 2 \cdot (-6)^2 = 0$ *so* $y'' = -36$

5. (10 points) Find all the points (x,y) on the curve

$$x = t^2 + 8t, \quad y = 2t + 3\ln t$$

where the tangent line is parallel to the line $x - 2y = 7$.

Rewrite the line $y = \frac{1}{2}x - \frac{7}{2}$ *to see that the slope is* $m = \frac{1}{2}$.

Calculate $\frac{dy}{dx}$ *and set it equal to* $\frac{1}{2}$.

$$\begin{aligned}
\frac{dx}{dt} &= 2t + 8 \\
\frac{dy}{dt} &= 2 + \frac{3}{t} \\
\frac{dy}{dx} &= \frac{2 + \frac{3}{t}}{2t + 8} \\
&= \frac{2t + 3}{2t^2 + 8t}
\end{aligned}$$

$$\begin{aligned}
\frac{2t + 3}{2t^2 + 8t} &= \frac{1}{2} \\
2t + 3 &= t^2 + 4t \\
0 &= t^2 + 2t - 3 \\
&= (t + 3)(t - 1) \\
t &= -3, 1
\end{aligned}$$

$x(1) = 9$ *and* $y(1) = 2$.

$x(-3) = -15$ *but* $y(-3)$ *is undefined.*

The only point is $(9,2)$