Math 124G Fall '19 MT2 Solutions

1. Find $\frac{dy}{dx} = y'$ for the following functions.

(a) $y = \frac{1}{x^2 + \sqrt{\tan x + 4^x}}$

$$y' = -\frac{2x + \frac{\sec^2 x + (\ln 4) 4^x}{2\sqrt{\tan x + 4^x}}}{\left(x^2 + \sqrt{\tan x + 4^x}\right)^2}$$

(b) $xe^y - 5xy + xy^2 = \cos y$

$$e^y + xe^y y' - 5y - 5xy' + y^2 + x\,2yy' = (-\sin y)\,y'$$

$$(xe^y - 5x + 2xy + \sin y)\,y' = -e^y + 5y - y^2$$

$$y' = \frac{-e^y + 5y - y^2}{xe^y - 5x + 2xy + \sin y}$$

(c) $y = (x^2+1)^{5x}$

$$\ln y = \ln (x^2+1)^{5x} = 5x \ln (x^2+1)$$

$$\frac{y'}{y} = 5 \ln(x^2+1) + \frac{5x(2x)}{x^2+1}$$

$$y' = \left(5 \ln(x^2+1) + \frac{10x^2}{x^2+1}\right)(x^2+1)^{5x}$$

2. Use linear approximation to estimate the value of $\sqrt[5]{33}$. Is your approximation more or less than the actual value. Explain. (This question must be done without the use of a calculator.)

$f(x) = x^{1/5}$ $\quad a = 32 \quad f(32) = 2$

$f'(x) = \frac{1}{5}x^{-4/5}$ $\quad f'(32) = \frac{1}{5(32)^{4/5}} = \frac{1}{80}$

Tangent Line

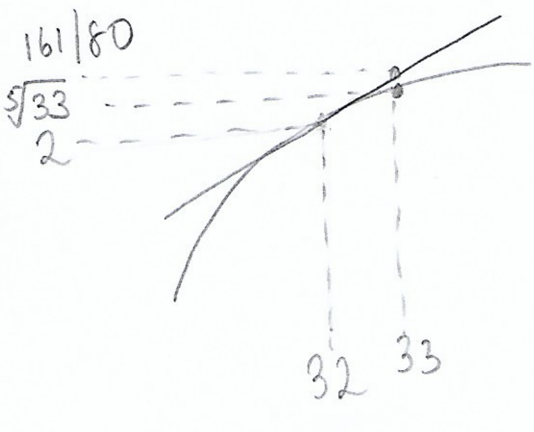
$y - 2 = \frac{1}{80}(x - 32)$

Linear Approximation

$y - 2 \simeq \frac{1}{80}(33 - 32)$

$y \simeq \frac{1}{80} + 2 = \frac{161}{80}$ $\quad$ so $\sqrt[5]{33} \simeq \frac{161}{80}$

$f'' = \frac{-4}{25}x^{-9/5}$ $\quad f''(32) = \frac{-4}{5(32)^{9/5}} < 0$



so $\frac{161}{80} > \sqrt[5]{33}$

3. Compute the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where $t = 2$ for the parametric equations

$$x = 5e^{t-2} + 5t^3 \qquad y = \ln(t-1) + t.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t-1} + 1}{5e^{t-2} + 15t^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{1+1}{5+15(4)} = \frac{2}{65}$$

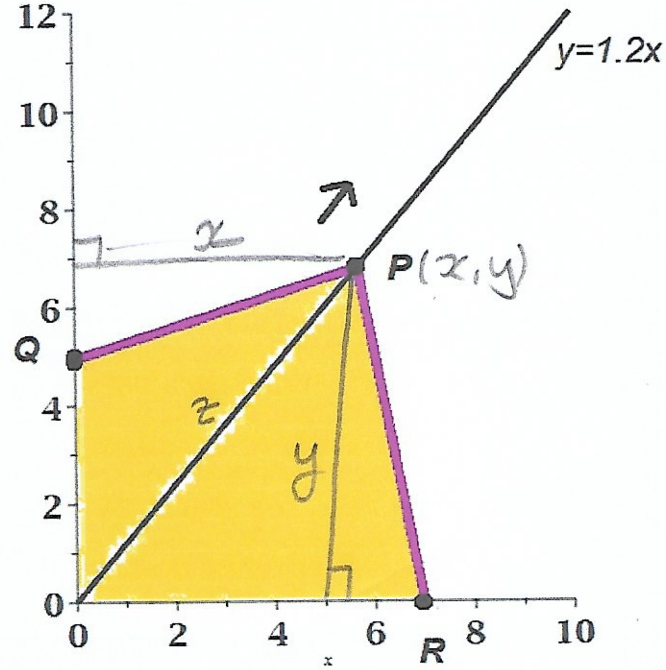
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{\dfrac{\frac{-1}{(t-1)^2}\left(5e^{t-2} + 15t^2\right) - \left(\frac{1}{t-1} + 1\right)\left(5e^{t-2} + 30t\right)}{\left(5e^{t-2} + 15t^2\right)^2}}{\left(5e^{t-2} + 15t^2\right)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{\dfrac{-(5+60)^2 - (2)(5+60)}{(5+60)^2}}{(5+60)} = \frac{-3(65)}{65^3} = -\frac{3}{65^2}$$

4. An elastic band is attached to the points $Q(0,5)$, $R(7,0)$, and P. The point P moves on the line $y = 1.2x$ getting away from the origin at a speed of 1.7 units per second.

(a) How fast is the x-coordinate of the point P changing?



$$z^2 = x^2 + y^2 = x^2 + (1.2x)^2$$
$$z^2 = 2.44x^2$$
$$z = \sqrt{2.44}\,x$$
$$1.7 = \frac{dz}{dt} = \sqrt{2.44}\,\frac{dx}{dt}$$
$$\frac{dx}{dt} = \frac{1.7}{\sqrt{2.44}}$$

(b) How fast is the shaded area increasing when the point P is 15 centimeters from the origin?

$$A = \frac{7y}{2} + \frac{5x}{2}$$ (areas of the triangles on the right and left)
$$= \frac{1}{2}\left(7(1.2x) + 5\right) = 6.7x$$
$$\frac{dA}{dt} = 6.7\frac{dx}{dt} = 6.7\left(\frac{1.7}{\sqrt{2.44}}\right) = \frac{11.39}{\sqrt{2.44}}$$