1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) (4 points) $\lim\limits_{t\to 1} \dfrac{2t^2+t-3}{t^2+4t-5}$

$$\begin{aligned}\lim_{t\to 1} \frac{2t^2+t-3}{t^2+4t-5} &= \lim_{t\to 1} \frac{(2t+3)(t-1)}{(t+5)(t-1)} \\ &= \lim_{t\to 1} \frac{2t+3}{t+5} \\ &= \frac{5}{6}\end{aligned}$$

   (b) (4 points) $\lim\limits_{t\to\infty} \cos\left(\dfrac{t+1}{1+3t-5t^2}\right)$

$$\begin{aligned}\lim_{t\to\infty} \cos\left(\frac{t+1}{1+3t-5t^2}\right) &= \cos\left(\lim_{t\to\infty} \frac{t+1}{1+3t-5t^2}\right) \\ &= \cos\left(\lim_{t\to\infty} \frac{\frac{1}{t}+\frac{1}{t^2}}{\frac{1}{t^2}+\frac{3}{t}-5}\right) \\ &= \cos\left(\frac{0}{-5}\right) \\ &= \cos(0) = 1\end{aligned}$$

   (c) (4 points) $\lim\limits_{x\to 3} \dfrac{\sqrt{2x-1}}{x^2-6x+9}$

   The numerator is approaching $\sqrt{5}$ *and the denominator is approaching* 0. *This is an infinite limit.*

   The numerator is never negative. The denominator equals $(x-3)^2$. *This is positive on either side of* $x=3$.

   Thus $\lim\limits_{x\to 3} \dfrac{\sqrt{2x-1}}{x^2-6x+9} = +\infty$

2. (7 points) Use the limit definition of the derivative on this problem. Do not use differentiation formaulas. Find the slope of the tangent line to the curve $y = \sqrt{x+3}$ at the point $(1,2)$.

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=1} &= \lim_{h\to 0} \frac{y(1+h) - y(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h\to 0} \frac{4+h-4}{h\cdot\left(\sqrt{4+h}+2\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} \\ &= \frac{1}{4} \end{aligned}$$

3. (7 points) Calculate the equation of the tangent line to $g(x) = (x+2)\,e^x$ at $x = 0$.

The equation of the tangent line is $y - b = m(x-a)$.

Here $a = 0$ *and* $b = g(0) = 2$.

The slope is $m = g'(0)$.

$g'(x) = 1\cdot e^x + (x+2)\cdot e^x = (x+3)e^x$, *by the Product Rule.*

So $m = g'(0) = 3$

and the line is $y - 2 = 3(x-0)$

4. (8 points) Let c be a constant and $f(x) = \begin{cases} x^2+2 & \text{if } x<1; \\ cx-5 & \text{if } x \geq 1. \end{cases}$

   Find the value of c that makes $f(x)$ a continuous function everywhere. Use limits to give a careful justification of your answer.

   First note that if $x \neq 1$ then $f(x)$ is continuous at x. Indeed, if $x<1$ then $f(x)$ is a quadratic polynomial and if $x>1$ then $f(x)$ is a linear polynomial.

   To make $f(x)$ continuous at $x=1$ we need $\lim_{x\to 1} f(x) = f(1)$.

   Note that $f(1) = c-5$.

   We need to look at the left and right limits.

   $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} cx-5 = c-5$*, so the function is continuous from the right for every value of c.*

   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+2 = 3$

   The function will be continuous at $x=1$ if the left limit equals the right limit: $3 = c-5$

   Thus the value $c=8$ will make $f(x)$ continuous everywhere.

5. (8 points) A particle is travelling in a straight line. Its position is given by $x = t^3 - 6t^2 + 12t$, where x is in feet and t is in seconds. Find the acceleration when the velocity is 0.

   First find the time(s) when the velocity is zero.

$$\begin{aligned} v = \frac{dx}{dt} &= 3t^2 - 12t + 12 \\ 0 &= 3t^2 - 12t + 12 \\ &= 3(t-2)^2 \\ t &= 2 \text{ seconds} \end{aligned}$$

   Now compute the acceleration at $t=2$.

$$\begin{aligned} a = \frac{dv}{dt} &= 6t - 12 \\ a(2) &= 0 \text{ ft/sec}^2 \end{aligned}$$

6. (8 points) Where does the normal line to the parabola $y = x^2 - 3x + 4$ at the point $(1,2)$ intersect the parabola a second time? Give both coordinates of the point of intersection.

First compute the normal line at $x = 1$.

$$\begin{aligned} \frac{dy}{dx} &= 2x - 3 \\ m_{\tan} &= \left.\frac{dy}{dx}\right|_{x=1} = -1 \end{aligned}$$

The slope of the normal is $m_\perp = \dfrac{-1}{m_{\tan}} = 1.$

The normal line is $y - 2 = 1 \cdot (x-1)$ *or* $y = x + 1.$

Now intersect the normal line and the parabola by eliminating y.

$$\begin{aligned} x + 1 &= x^2 - 3x + 4 \\ 0 &= x^2 - 4x + 3 \\ &= (x-1)(x-3) \\ x &= 1, 3 \end{aligned}$$

$x = 3$ *is the value we are looking for. At this value* $y = 3 + 1 = 4.$

The point is $(3,4)$.