1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) (4 points) $\lim_{x\to\infty} \sqrt[3]{\frac{1-x^2}{8x^2-5x}}$

$$\begin{aligned}
\lim_{x\to\infty} \sqrt[3]{\frac{1-x^2}{8x^2-5x}} &= \sqrt[3]{\lim_{x\to\infty} \frac{1-x^2}{8x^2-5x}} \qquad \textit{because } f(x)=\sqrt[3]{x} \textit{ is continuous} \\
&= \sqrt[3]{\lim_{x\to\infty} \frac{1-x^2}{8x^2-5x}\cdot\frac{1/x^2}{1/x^2}} \\
&= \sqrt[3]{\lim_{x\to\infty} \frac{1/x^2-1}{8-5/x}} \\
&= \sqrt[3]{\frac{-1}{8}} \\
&= -\frac{1}{2}
\end{aligned}$$

   (b) (4 points) $\lim_{x\to 2} \frac{x-\sqrt{x+2}}{x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{x-\sqrt{x+2}}{x-2} &= \lim_{x\to 2} \frac{x-\sqrt{x+2}}{x-2}\cdot\frac{x+\sqrt{x+2}}{x+\sqrt{x+2}} \\
&= \lim_{x\to 2} \frac{x^2-x-2}{(x-2)(x+\sqrt{x+2})} \\
&= \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x+\sqrt{x+2})} \\
&= \lim_{x\to 2} \frac{x+1}{x+\sqrt{x+2}} \\
&= \frac{3}{4}
\end{aligned}$$

   (c) (4 points) $\lim_{x\to 2} \frac{2x^2-x-5}{x^3-4x^2+12}$

   This function is continuous at $x=2$.

$$\lim_{x\to 2} \frac{2x^2-x-5}{x^3-4x^2+12} = \frac{1}{4}$$

2. (8 points) Use the limit definition of the derivative on this problem. Do not use differentiation formaulas. Find the slope of the tangent line to the curve $y = \frac{5}{x^2+1}$ at the point $(2,1)$.

$$
\begin{aligned}
\left.\frac{dy}{dx}\right|_{x=2} &= \lim_{h\to 0} \frac{\frac{5}{(2+h)^2+1} - 1}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{5 - \big((2+h)^2+1\big)}{(2+h)^2+1} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-4h - h^2}{(2+h)^2+1} \\
&= \lim_{h\to 0} \frac{-4-h}{(2+h)^2+1} \\
&= -\frac{4}{5}
\end{aligned}
$$

3. (7 points) Calculate the equation of the tangent line to $f(t) = 3t^2 \cdot \cos(t)$ at $t = \pi$.

$f(\pi) = -3\pi^2$

$f'(t) = 6t\cos t - 3t^2 \sin t$

$f'(\pi) = -6\pi$

The tangent line is $y + 3\pi^2 = -6\pi(t - \pi)$

4. (8 points) Let c be a constant and $f(x) = \begin{cases} x^2 - 2x + c & \text{if } x \leq 2; \\ cx + 5 & \text{if } x > 2. \end{cases}$

   Find all values of c that make $f(x)$ a continuous function. If there are none, explain why. Use limits to give a careful justification of your answer.

   If $a \neq 2$ then $f(x)$ is continuous at $x = a$. Indeed, if $a > 2$ then $f(x)$ is a linear polynomial near a. If $a < 2$ then $f(x)$ is a quadratic polynomial near a. All polynomials are continuous.

   At $x = 1$ we must look at the left and right limits.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 1^-} x^2 - 2x + c \\ &= c \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 1^+} cx + 5 \\ &= 2c + 5 \end{aligned}$$

   For $f(x)$ to be continuous at $x = 2$ we need $c = 2c + 5$. *The solution is $c = -5$.*

5. (7 points) Find all the points (a, b) on the curve $y = \left(x^2 - 3\right) e^x$ where the tangent line is horizontal.

   First, calculate $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= 2x\,e^x + \left(x^2 - 3\right) e^x \\ &= \left(x^2 + 2x - 3\right) e^x \end{aligned}$$

   Next, set $\dfrac{dy}{dx} = 0$ *and solve for x.*

$$\begin{aligned} 0 &= \left(x^2 + 2x - 3\right) e^x \\ &= (x+3)(x-1)e^x \end{aligned}$$

   Thus $x = -3, 1$. The points are $\left(-3, 6/e^3\right)$ and $(1, -2e)$

6. (8 points) Where does the tangent line to the curve $y = x^3 - 2x^2 + x$ at the point where $x = 0$ intersect the curve a second time? Give both coordinates of the point of intersection.

First compute the equation of the tangent line.

$x = 0$

$y = 0^3 - 2 \cdot 0 + 0 = 0$

$\frac{dy}{dx} = 3x^2 - 4x + 1 \qquad \left.\frac{dy}{dx}\right|_{x=0} = 1$

The tangent line is $y - 0 = 1 \cdot (x - 0)$, *or* $y = x$.

Now use elimination to intersect the line with the cubic curve.

$$\begin{aligned} x &= x^3 - 2x^2 + x \\ 0 &= x^3 - 2x^2 \\ &= x^2(x-2) \end{aligned}$$

The solutions are $x = 0, 2$. *The new value is* $x = 2$.

At $x = 2$ *we compute* $y = 2$ *(we are on the line* $y = x$*).*

The point of intersection is $(2, 2)$.