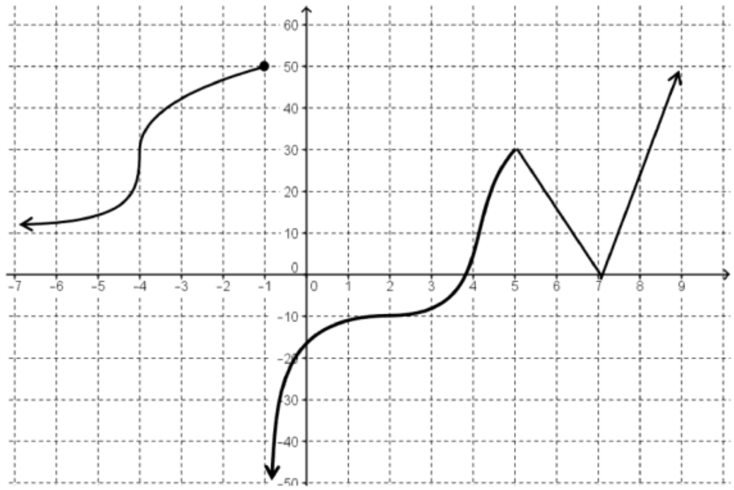
Mid 1 Sol

1. (11 points) The graph of a function $y = f(x)$ is shown below. Use it to answer the questions on this page. In this problem you do not need to show work, except as indicated.



(a) Does this graph have any **vertical** asymptotes? Circle YES or NO.
If yes, list their equation(s): $\boxed{x = -1}$

(b) Does this graph have any **vertical tangent lines**? Circle YES or NO.
If yes, list their equation(s): $\boxed{x = -4}$

(c) List all the values a where $f(x)$ is not continuous at $x = a$: $\boxed{a = -1}$

(d) List all the values a where $f(x)$ is not differentiable at $x = a$: $\boxed{a = -4, -1, 5, 7}$

(e) Evaluate the following limits, or state that the limit does not exist:

$\lim_{x\to 5} f(x) = \boxed{30}$

$\lim_{x\to(-1)^+} f(x) = \boxed{-\infty}$

$\lim_{x\to 5^+} \frac{f(x)-30}{x-5} = \lim_{x\to 5^+} f'(x) = \frac{-30}{2} = \boxed{-15}$ (slope on the right)

(f) The average rate of change of $f(x)$ between $x = 2$ and $x = 5$ is: $\frac{30-(-10)}{5-2} = \boxed{\frac{40}{3}}$

(g) The instantaneous rate of change of $f(x)$ at $x = 2$ is: $\boxed{0}$ (slope of tangent)

2. (10 points) Determine the values of the following limits, or state that the limit does not exist. If it is correct to say that the limit is $+\infty$ or $-\infty$, then you should say so. Show correct work or justification.

(a) $\lim_{x\to 0} \dfrac{\cos(x)-2}{\sin^2(x)}$ $= \boxed{-\infty}$ since numerator $\to -1$ & denominator $\to 0^+$

(b) $\lim_{x\to 0} \dfrac{\sin^2(\pi x)}{x^2}$ $= \lim_{x\to 0}\left(\dfrac{\sin^2(\pi x)}{\pi^2 x}\cdot \pi^2\right)$

$= \pi^2\left(\lim_{x\to 0}\dfrac{\sin(\pi x)}{\pi x}\right)^2 = \pi^2(1) = \boxed{\pi^2}$

(c) $\lim_{x\to 3} \dfrac{x^2-18x+45}{2x-6}$ $= \lim_{x\to 3}\dfrac{\cancel{(x-3)}(x-15)}{2\cancel{(x-3)}} = \dfrac{-12}{2} = \boxed{-6}$

3. (8 points) Compute the appropriate limits to determine if the function

$$f(x) = \sqrt{9x^2 - x + 1} - 3x$$

has any **horizontal asymptotes**. If yes, list (and box) their equations $y = b$. If there are none, state so as your answer. Show all work and use correct limit notation.

① As $x \to \infty$: $\lim_{x \to \infty} f(x)$ is indeterminate, of type $\infty - \infty$

$$\lim_{x \to \infty} \left(\sqrt{9x^2-x+1} - 3x\right) \cdot \frac{\sqrt{9x^2-x+1} + 3x}{\sqrt{9x^2-x+1} + 3x}$$

$$= \lim_{x \to \infty} \frac{\cancel{9x^2} - x + 1 - \cancel{9x^2}}{\sqrt{9x^2-x+1} + 3x} = \lim_{x \to \infty} \frac{-x+1}{\sqrt{9x^2-x+1} + 3x}$$

$$= \lim_{x \to \infty} \frac{-1 + 1/x}{\sqrt{9 - \frac{1}{x} + \frac{1}{x^2}} + 3} = \frac{-1+0}{\sqrt{9}+3} = -\frac{1}{6}$$

Hence $\boxed{y = -\frac{1}{6} \text{ is a horizontal asymptote as } x \to \infty}$

② As $x \to -\infty$:

$\lim_{x \to -\infty} \left(\sqrt{9x^2-x+1} - 3x\right)$ is of the type $\infty - (-\infty) = +\infty$

so <u>the graph has no horizontal asymptote as $x \to -\infty$.</u>

4. (5 points) Compute the derivative $g'(x)$ of the function $g(x) = (x+\cos x)e^x + 2x\sqrt{x} + \frac{\pi}{2}$.

$$g(x) = \underbrace{(x+\cos x)\,e^x}_{\text{PRODUCT RULE}} + 2\underbrace{x^{3/2}}_{\text{Power}} + \underbrace{\pi/2}_{\text{Constant Alone}}$$

$$g'(x) = (1-\sin x)\,e^x + (x+\cos x)\,e^x + \cancel{2}\,\frac{3}{\cancel{2}}\,x^{1/2}$$ ← OK to leave in this form

$$= \boxed{e^x\,(1+x-\sin x+\cos x) + 3\sqrt{x}}$$ ← nicer form

5. (8 points) Given the function:

$$f(x) = \begin{cases} cx^2 - x + 3, & \text{for } x \leq -1 \\ x + d, & \text{for } x > -1 \end{cases}$$

(a) Find an equation satisfied by the constants c and d if the function f is continuous at $x = -1$.

f is continuous at $x=-1$: $\lim_{x\to(-1)^-} f(x) = f(-1) = \lim_{x\to(-1)^+} f(x)$

so if & only if $c(-1)^2 - (-1) + 3 = (-1) + d$

i.e $c + 4 = -1 + d$, or $\boxed{d = c+5}$

(b) Compute the values c and d for which the function f is also differentiable at $x = -1$.

To avoid a cusp at $x=-1$, we need to have the same slope on both sides of $x=-1$, i.e.

$\lim_{x\to(-1)^-} f'(x) = \lim_{x\to(-1)^-} (2cx-1) = -2c-1$

must equal $\lim_{x\to(-1)^+} f'(x) = 1$

$-2c-1 = 1$

$2c = -2$

$\boxed{c=-1}$

& $d = c+5$

so $\boxed{d=4}$.

6. (8 points) Find the equations of all the tangent lines to the curve:

$$y = \frac{x-1}{x+1}$$

which are parallel to the line

$$x - 2y = 3. \Rightarrow 2y = x + 3 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$$

Show your work and put your answers in the form $y = mx + b$.

Using Quotient Rule: $\frac{dy}{dx} = \frac{(x-1)'(x+1) - (x-1)(x+1)'}{(x+1)^2} = \frac{\cancel{x}+1-\cancel{x}+1}{(x+1)^2}$

$$\frac{dy}{dx} = \frac{2}{(x+1)^2}$$

The tangent line to $y = \frac{x-1}{x+1}$ at $x = a$ is parallel to the given line if its slope is $\frac{1}{2}$ i.e.

$$\frac{2}{(a+1)^2} = \frac{1}{2}$$

$$(a+1)^2 = 4$$

$$a+1 = \pm 2$$

$$a = -1 \pm 2$$

1) $\underline{a = -3}$: $f(a) = \frac{-3-1}{-3+1} = \frac{-4}{-2} = 2$

tan line: $y = \frac{1}{2}(x+3) + 2 \Rightarrow \boxed{y = \frac{1}{2}x + \frac{7}{2}}$

2) $\underline{a = 1:}$ $f(a) = \frac{1-1}{1+1} = 0$

tangent line: $y = \frac{1}{2}(x-1) + 0 \Rightarrow \boxed{y = \frac{1}{2}x - \frac{1}{2}}$