Midterm Key

Sunday, January 30, 2022 9:37 AM

Math 124 **Midterm I, February 1st** **Winter 2022**

HONOR STATEMENT
I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam.

Name	Signature

Student ID #

1.	2.	3.	4.	5.	6.	7.	8.	$\sum$
10	10	10	10	10	10	10	10	80

- You have 80 minutes for 8 problems. Check your copy of the exam for completeness.
- You are allowed to use a hand written sheet of paper (8x11 in), back and front.
- Calculators may only have basic functions, but no graphing or differentiation functions.
- Justify all your answers and show your work for credit.
- All answers must be exact, no rounding.
- Credit given for formal aspects

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

Problem 1. Find the limit of the following expression. Your answer must be a real number, ∞, $-\infty$, or DNE, whatever fits **best**. Justify all your work. No rounding

$$\lim_{x\to 6} \frac{x^2-4x-12}{x^2-5x-6}.$$

$$\lim_{x\to 6} \frac{x^2-4x-12}{x^2-6x-6} \overset{\boxed{\frac{0}{0}}}{=} \lim_{x\to 6} \frac{(x-6)(x+2)}{(x-6)(x+1)} = \lim_{x\to 6} \frac{x+2}{x+1} \overset{\text{DSP}}{=} \frac{8}{7}$$

Problem 2. Find the limit of the following expression. Your answer must be a real number, ∞, $-\infty$, or DNE, whatever fits best. Justify all your work.

$$\lim_{x\to 3} \arctan\left(\frac{1}{(x-3)^2}\right).$$

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = \left[\frac{C^{+}}{0^{+}}\right]$$

So $\displaystyle \lim_{x\to 3} \arctan\left(\frac{1}{(x-3)^2}\right) = \lim_{z\to\infty} \arctan(z) = \frac{\pi}{2}$

Problem 3. The function

$$f(x) = \begin{cases} 12 - 2\sqrt{x} & , \quad x \geq 1 \\ ax^2 - a + 10 & , \quad x < 1 \end{cases}$$

is continuous at $x = 1$ (you do not need to show this). Determine the constant a so that f is differentiable at all places in $\mathbb{R}$.

$12-2\sqrt{x}$ is diff for all $x>1$ (lecture-fact about domains)

ax^2-a+10 is diff for all $x<1$

Check transition.

From the left:

$x>1$: $f'(x) = -\frac{1}{\sqrt{x}}$ $\quad \lim_{x\to 1^+} f'(x) = -\frac{1}{1} = -1$

$x<1$: $f'(x) = 2x \cdot a$ $\quad \lim_{x\to 1^-} f'(x) = 2a$

so $2a = -1 \iff \boxed{a = -\frac{1}{2}}$

Problem 4. Use the definition of a derivative (the limit-version, no differentiation rules) to find the derivative function of $f(x) = \frac{1}{\sqrt{x}}$ at $x=3$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{3+h}} - \frac{1}{\sqrt{3}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{3} - \sqrt{3+h}}{\sqrt{3}\sqrt{3+h}}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{3} - \sqrt{3+h})}{h\cdot\sqrt{3}\cdot\sqrt{3+h}} \cdot \frac{(\sqrt{3}+\sqrt{3+h})}{(\sqrt{3}+\sqrt{3+h})}$$

$$= \lim_{h\to 0} \frac{\cancel{3} - \cancel{3} - h}{h\cdot\sqrt{3}\sqrt{3+h}\cdot(\sqrt{3}+\sqrt{3+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{3}\sqrt{3+h}\cdot(\sqrt{3}+\sqrt{3+h})}$$

$$\overset{DSP}{=} \frac{-1}{\sqrt{3}\cdot\sqrt{3}(\sqrt{3}+\sqrt{3})} = -\frac{1}{3\cdot 2\sqrt{3}} = -\frac{1}{6\sqrt{3}}$$

Problem 5. Find the points on the curve

$$y = 2x^3 + 3x^2 - 12x + 1$$

where the tangent line is horizontal.

$y' = 6x^2 + 6x - 12$

horizontal, when $y' = 0$

$6x^2 + 6x - 12 = 0$

$x^2 + x - 2 = 0$

$(x + \frac{1}{2})^2 = 2 + \frac{1}{4}$

$(x + \frac{1}{2})^2 = \frac{9}{4}$

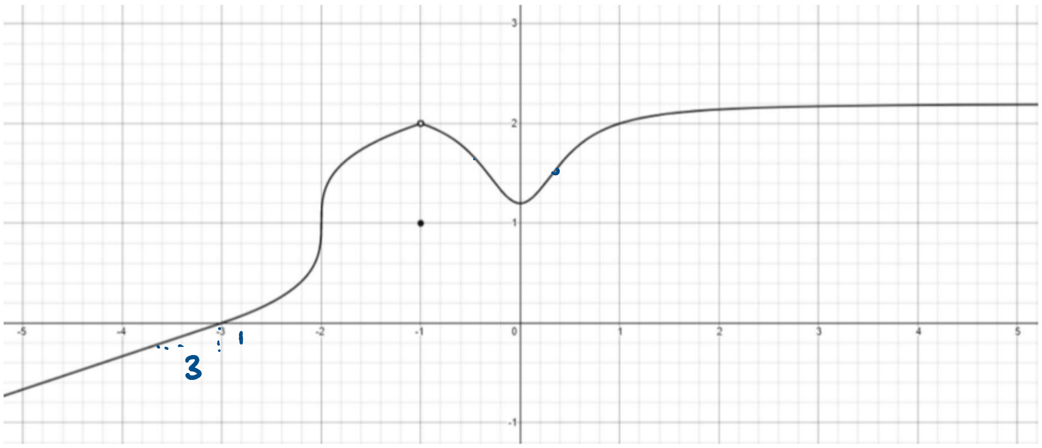
$x = -\frac{1}{2} + \frac{3}{2}$ OR $x = -\frac{1}{2} - \frac{3}{2}$

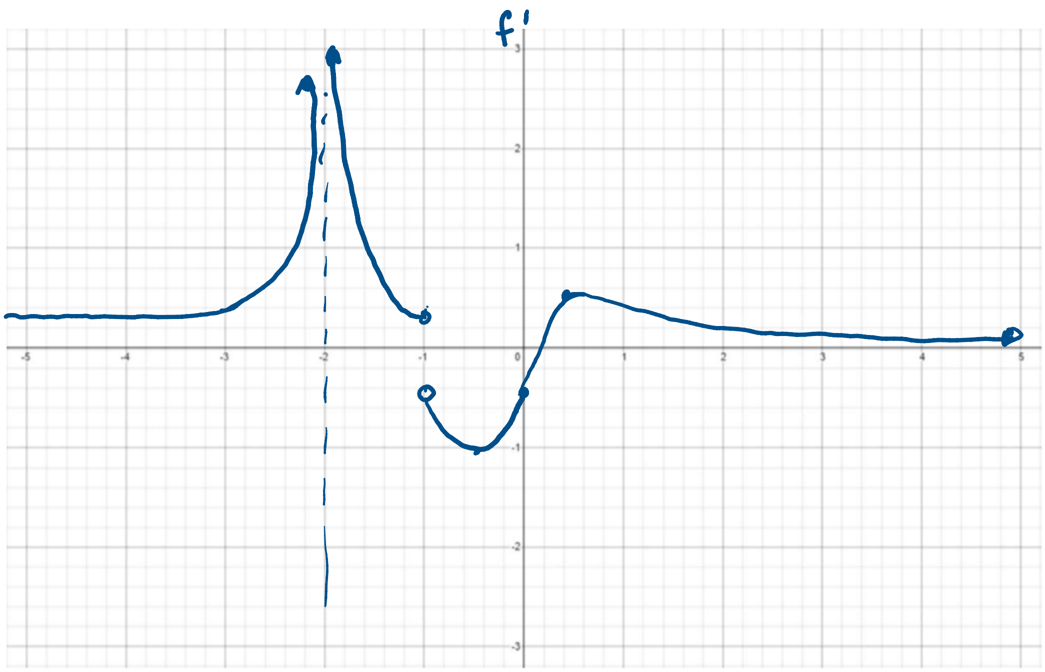
$x = 1$ OR $x = -2$

$\boxed{(1, -6) \qquad (-2, 21)}$

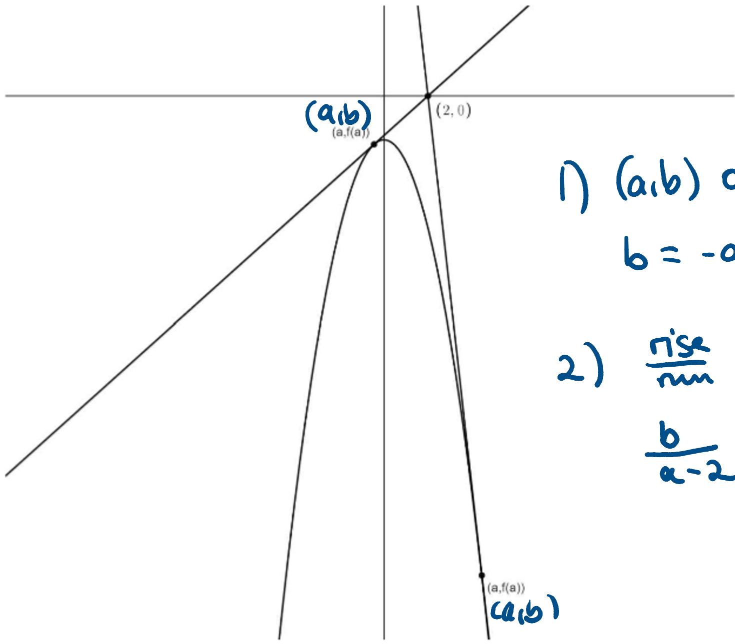
Problem 6. Consider the following graph of the function $f(x)$. In the given blank coordinate system sketch the graph of $f'(x)$. Be sure to correctly sketch

- Where $f'(x)$ is positive/negative or equal to 0.
- Where $f'(x)$ is constant and what value that constant is.
- Where $f'(x)$ is increasing and where it is decreasing.
- Where $f'(x)$ is not defined.





Problem 7. Consider the function $f(x) = -x^2 - 2$. Find the values both values for a so that the tangent lines through $(a, f(a))$ to the graph of f pass through the point $(1, 0)$ as shown in the sketch below.



1) (a,b) on curve:

$$b = -a^2 - 2$$

2) $\frac{\text{rise}}{\text{run}} = f'(a)$ $\qquad f'(x) = -2x$

$$\frac{b}{a-2} = -2a$$

combine: $\frac{-a^2-2}{a-2} = -2a \quad \begin{array}{l} |\cdot(-1) \\ \cdot(a-2) \end{array}$

$$a^2+2 = 2a(a-2)$$

$$a^2+2 = 2a^2-4a$$

$$2 = a^2-4a$$

$$2+4 = (a-2)^2$$

$$\pm\sqrt{6} = a-2$$

$$a = 2-\sqrt{6} \quad \text{OR} \quad a = 2+\sqrt{6}$$

Problem 8. The population of bacteria in a petri dish can be modeled by

$$N(t) = \frac{\overset{3250}{\cancel{3}}t + 1}{t+1},$$

⊛

where t measures the time in days ($t \geq 0$). Show these **TWO** features of the population:

(a) The population of the bacteria is always (for all $t \geq 0$) growing but/and

(b) The number of bacteria ~~is bounded, even if we observed forever.~~ ⊛ is nearing a certain number when we observe for a very long time.

(a) $$N'(t) = \frac{3250(t+1) - (3250t+1)}{(t+1)^2}$$

$N'(t) = \frac{3249}{(t+1)^2}$ We note that $N'(t) > 0$ f.a. t's, which means that $N(t)$ is increasing for all t.

(b) We observe what happens in the far future by considering

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{\frac{1}{t}(3250t+1)}{\frac{1}{t}(t+1)} = \lim_{t\to\infty} \frac{3250 + \cancel{\frac{1}{t}}^{\,0}}{1 + \cancel{\frac{1}{t}}^{\,0}} = 3250.$$

Because $y = 3250$, the statement is true.