Math 124B, Spring 2022 Midterm I

April 26, 2022

Name KEY

Student Number

Instructions.

- **These exams will be scanned. Please write your name and student number clearly for easy recognition.**
- There are 5 questions. The exam is out of 50 points.
- You are allowed to use one page of handwritten notes, 8.5 x 11, both sides ok.
- You can only use a Ti-30x IIS calculator. Unless otherwise stated, you have to give exact answers to questions. ($\frac{2\ln 3}{\pi}$ and 1/3 are exact, 0.699 and 0.333 are approximations for the those numbers.)
- Each problem clearly states if you must show work. In cases where work is requested, **you may not get full credit for a right answer if your answer is not justified by your work.**

Question	points	Score
1	~~12~~ 13	
2	12	
3	11	
4	8	
5	6	
Total	50	

1. (1~~2~~3 points) On this problem, place the answers you want graded in the provided boxes. If a box is left blank, you will get 0 points on that question. Give EXACT answers. Determine if the limit exists as a finite number or $\pm\infty$ or DNE (does not exist). No work required. Some answers may involve unknown constants, as specified in each question. ~~in (a) – (d)~~ unless requested.

2pt (a) $\lim_{h\to 5}\left(\frac{h-5}{h^2-25}\right) = \boxed{\frac{1}{10}}$

$\lim_{h\to 5}\frac{\cancel{(h-5)}}{\cancel{(h-5)}(h+5)} = \lim_{h\to 5}\frac{1}{h+5} = \frac{1}{10}$

2pt (b) $\lim_{h\to\infty}\left(\frac{h-5}{h^2-25}\right) = \boxed{0}$

$\lim_{h\to\infty}\frac{\frac{h}{h}-\frac{5}{h}}{\frac{h^2}{h}-\frac{25}{h}} = \lim_{h\to\infty}\frac{1-\frac{5}{h}}{h-\frac{25}{h}} = 0$

(c) If A and B are non-zero constants,

2pt $\lim_{t\to\frac{\pi}{2}}\left(\frac{\sqrt{\sin^2 t + A\cos^2 t} - \sin t}{B\cos^2 t}\right) = \boxed{\frac{A}{2B}}$

$\frac{s^2+Ac^2-s^2}{Bc^2(\sqrt{s^2+Ac^2}+s)} = \frac{A}{B(\sqrt{s^2+Ac^2}+s)} = \frac{A}{B(\sqrt{1+0}+1)} = \frac{A}{2B}$

(d) Consider the multipart function $f(x) = \begin{cases} 3x^2 & \text{if } x \le 0 \\ -4x^2+1 & \text{if } x > 0 \end{cases}$

$f' = \begin{cases} 6x & x<0 \\ -8x & x>0 \end{cases}$

(i)

2pt $\lim_{h\to 0^-} f'(h) = \boxed{0}$

$\lim_{h\to 0^-} 6h = 0$

(ii)

2pt $\lim_{h\to 0^+} f'(h) = \boxed{0}$

$\lim_{h\to 0^+} -8h = 0$

(iii) Is $f(x)$ differentiable at $x = 0$? Explain. No

1 pt NO

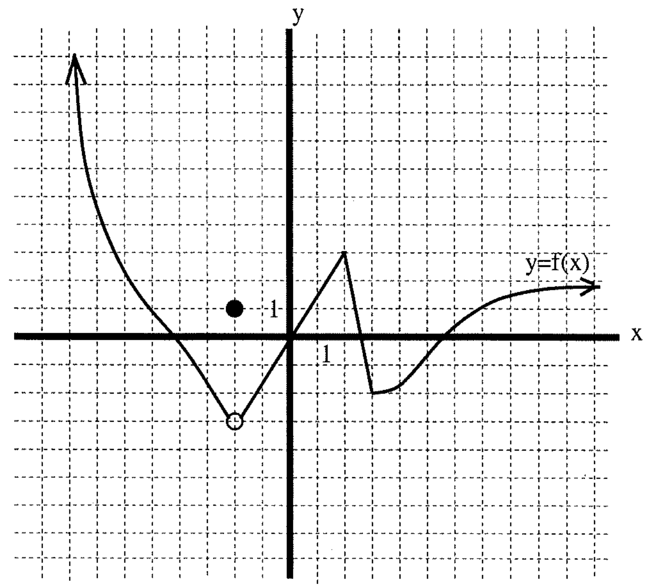
2pt valid reason

$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{-4h^2+1}{h} = \infty$

$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{3h^2}{h} = \lim_{h\to 0^-} 3h = 0$

$\Rightarrow f'(0)$ DNE since right + left limits not equal

2. (12 points) The graph of a function $y = f(x)$ is pictured. The x and y axis are indicated and the dotted lines yield a grid with units of 1 in each direction. No work required on this part. We will only grade your BOXED final answers. If you are asked to calculate a limit, determine if the limit exists as a finite number or $\pm\infty$ or DNE (does not exist). No work required on this problem.



1pt (a) $\lim_{x \to -2} f(x) = \boxed{-3}$

1pt (b) $\lim_{x \to 0} \frac{f(x)}{x} = \boxed{3/2}$ $= \lim_{h \to 0} \frac{f(h+0) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = f'(0) =$ slope line $= 3/2$

1pt (c) Is $y = \cos(\pi/x) \cdot f(x)$ is continuous at $x = -2$? $\boxed{\text{Yes.}}$

1) $y(-2) = \cos(-\frac{\pi}{2}) f(-2) = 0 \cdot 1 = 0$

2) $\lim_{x \to -2} y(x) = 0 \cdot (-3) = 0$

(d) What is the slope of the tangent line to the curve $y = \cancel{(x + e^x) f(x)}$ $\frac{f(x)}{e^x}$ at ~~$x = 2$?~~ $x = -1$?

2pt $\boxed{3e}$

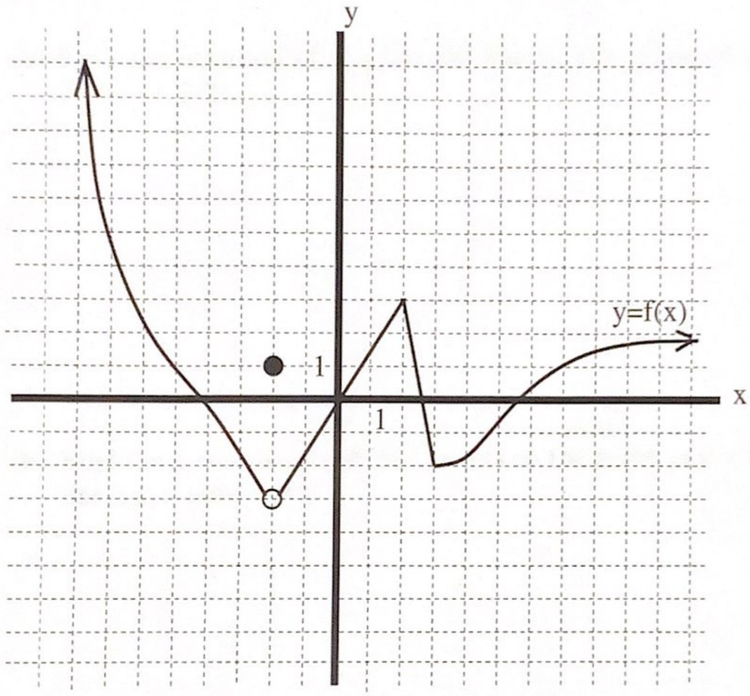
$y'(-1) = \frac{f'(-1) - f(-1)}{e^{-1}} = \frac{3/2 - (-3/2)}{e^{-1}} = \frac{6}{2e^{-1}} = \frac{6e}{2} = 3e$

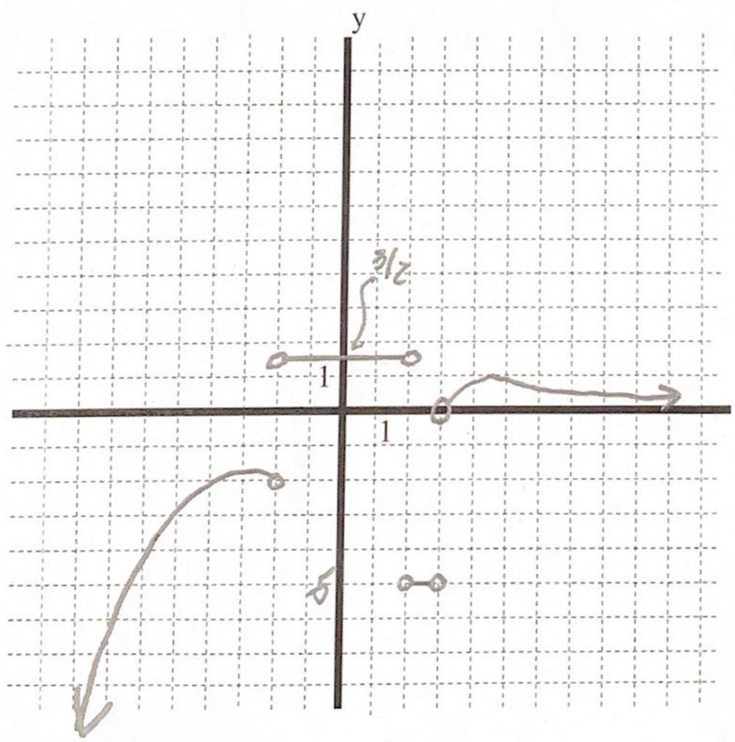
$y' = \frac{f'(x) - f(x)}{e^x} = \frac{e^x f'(x) - f(x) e^x}{e^x e^x} = y'$

1pt (e) Circle the LARGEST number in this list:

$f(0)$ $= 0$ $\quad$ $f'(0)$ $3/2$ $\quad$ $f''(0)$ 0

2. (continued). The graph of $f(x)$ is reproduced here. Sketch the graph of $y = f'(x)$ on the grid below:





Ⓐ: Must go to $-\infty$ as $x \to -\infty$ and always negative

Ⓑ: constant $y = \frac{3}{2}$

Ⓒ: constant $y = -5$

Ⓓ: starts close to 0 at $x = 3$, then always positive with local max as $\to 0$ as $x \to \infty$.

A (2pt) B (1pt) C (1pt) D (2pt)

3. ($\cancel{12}$ 11 points) Suppose $d(x) = \dfrac{1}{1+x^2}$.

(a) Find the equation of the tangent line to the graph of $y = d(x)$ at the point $P = (-1, 1/2)$.

$\cancel{2pt}$ $\longrightarrow$ $d'(x) \underset{QR}{=} \dfrac{(1+x^2)\cdot 0 - 1\cdot 2x}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$) given

4 $\cancel{3pt}$

T. Line @ P: $y = (\text{slope @ } P)(x-(-1)) + \frac{1}{2}$

(2) $\rightarrow$ $= d'(-1)(x+1) + \frac{1}{2}$

(2) $\rightarrow$ $y = \dfrac{-2(-1)}{(1+(-1)^2)^2}(x+1) + \frac{1}{2} = \frac{1}{2}(x+1) + \frac{1}{2}$ $= \frac{1}{2}x + 1$

(b) Find the x-coordinates of ALL points on the graph of $y = d(x)$ where the tangent line has x-intercept 2.

- Let $\underset{= Q}{(a, d(a))}$ be pt on graph where T. Line has x-intercept 2.
- T. Line eqn @ Q:

$$y = d'(a)(x-a) + d(a)$$

2 $\cancel{3pt}$ $\Rightarrow$ (*) $\boxed{y = \dfrac{-2a}{(1+a^2)^2}(x-a) + \dfrac{1}{1+a^2}}$

3pt

- Impose condition T. Line thru (2,0): set $y=0, x=2$ in (*)

$$0 = \frac{-2a}{(1+a^2)^2}(2-a) + \frac{1}{1+a^2}$$

$$\frac{1}{1+a^2} = \frac{2a}{(1+a^2)^2}(2-a) \Rightarrow 1+a^2 = 2a(2-a)$$

$$\Rightarrow 1+a^2 = 4a - 2a^2 \Rightarrow \boxed{3a^2 - 4a + 1 = 0}$$

- Apply Quad formula to solve for a:

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(+1)}}{2\cdot 3} = \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6}$$

2pt

$$a = \frac{2 \pm 1}{3} \Rightarrow \boxed{a = 1 \text{ or } a = \tfrac{1}{3}}$$

1 pt each value correct

4. (8 points) An object moves along the x-axis and its location at time $t \geq 0$ seconds is given by the function

$$x(t) = 10 - \frac{50t}{9 + t^2}$$

where is object located when the velocity is 0?

and units on the axis are "meters". Assume $t \geq 0$. ~~On the time interval [0, 10], determine ALL times when the object is moving to the right.~~ You must show work and explain how you got your answer.

4pt for derivative using Q. Rule

$$x'(t) = \left(10 - \frac{50t}{9+t^2}\right)' = -\left(\frac{50t}{9+t^2}\right)'$$

$$\underset{QR}{=} -\left(\frac{(9+t^2)50 - 50t \cdot 2t}{(9+t^2)^2}\right)$$

$$= -\left(\frac{450 + 50t^2 - 100t^2}{(9+t^2)^2}\right)$$

$$= \frac{50t^2 - 450}{(9+t^2)^2} = 50\left(\frac{t^2 - 9}{(9+t^2)^2}\right) = 50\left(\frac{(t-3)(t+3)}{(9+t^2)^2}\right)$$

2pt

Now, $x'(t) = 0 \iff t = \pm 3$.

But $t > 0$, so $x'(t) = 0 \Rightarrow t = 3$

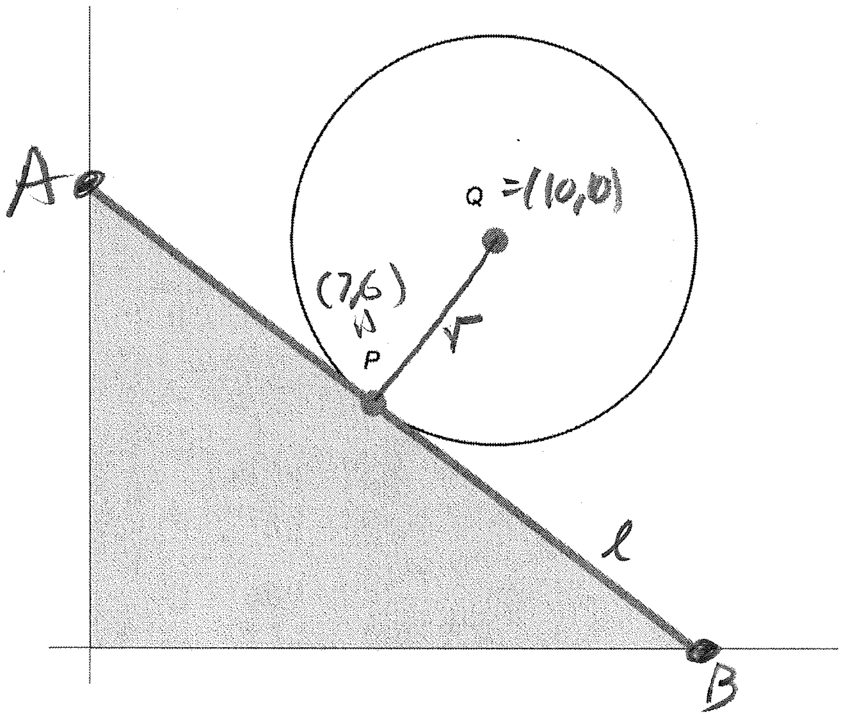
$$\Rightarrow x(3) = 10 - \frac{50 \cdot 3}{9 + 3^2} = 10 - \frac{150}{18}$$

$$= \frac{180 - 150}{18} = \frac{30}{18} = \frac{15}{9}$$

2pt

i.e. $\boxed{\text{object located @ } \frac{15}{9} \text{ when velocity is zero.}}$

5. (6 points) In the picture below is a circle of radius 5 centered at $Q = (10, 10)$. We have drawn the line ℓ tangent to the circle through the point $P = (7, 6)$. The line ℓ, the positive x-axis and the postive y-axis determine a triangular region, as pictured. What is the area of the triangular region? (You must explain how you arrived at your answer. No credit for answer only.)



$$\text{slope } \ell = \frac{-1}{\text{slope } r} = \frac{-1}{\frac{10-6}{10-7}} = \frac{-1}{\frac{4}{3}} = -3/4$$

(2pt) < eqn ℓ: $y = -3/4(x-7)+6$. (*)

(1pt) $\boxed{\text{Area} = \frac{1}{2} A \cdot B.}$

(1pt) < A: y-intercept: plug $x=0$ into eqn *: $y = -3/4(0-7)+6 = \frac{21}{4} + 6 = \frac{21+24}{4} = \frac{45}{4}$

(1pt) < B: x-intercept: plug $y=0$, solve for x in *: $0 = -3/4(x-7)+6 \Rightarrow -6 = -3/4(x-7)$, $8 = x-7 \Rightarrow \boxed{x=15}$

(1pt) < ∴ Area $= \frac{1}{2} \cdot \frac{45}{4} \cdot 15 = \frac{675}{8} = 84.375$

SCRATCH PAPER-DO NOT REMOVE