Math 124D – Midterm I

Autumn 2021

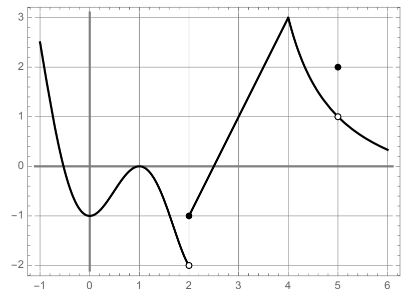
Solutions

Name: ____________________________

Student ID #: ____________________________

Question	Points	Score
1	12	
2	12	
3	12	
4	5	
5	6	
6	8	
Total:	55	

1. (12 points) Use the graph of the function $y = f(x)$ shown below to answer (a) – (f). For limit questions, your answer must be one of DNE, ∞, $-\infty$ or a number.



(a) Find $\lim_{x\to 2^+} f(x)$.

Solution: -1

(b) At what values of x is the derivative of f undefined?

Solution: 2, 4, 5

(c) List the intervals of values of x for which $f'(x) > 0$.

Solution: $(0, 1)$ and $(2, 4)$

(d) Find the derivative of $f(x)$ at $x = 3$.

Solution: $f'(3) = \frac{3-(-1)}{4-2} = 2$

(e) Find the derivative of $\dfrac{1}{f(x)}$ at $x = 3$.

Solution: $\left(\dfrac{1}{f(x)}\right)' = \dfrac{f(x)\cdot 0 - 1\cdot f'(x)}{(f(x))^2} = \dfrac{-f'(x)}{(f(x))^2} \quad \Rightarrow \quad \dfrac{-f'(3)}{(f(3))^2} = \dfrac{-2}{1^2} = -2$

(f) What is $\lim_{x\to 1} \dfrac{1}{f(x)}$?

Solution: $\lim_{x\to 1} f(x) = 0$ and $\frac{1}{f(x)} \leq 0$ for x near $1 \Rightarrow \lim_{x\to 1} f(x) = -\infty$

2. (12 points) A train leaves Station A at $t = 0$ along tracks that run straight east-west. Its distance east (in miles) of Station A t hours after departure is given by

$$d(t) = 80t^2 - 50t^4.$$

Make sure to **include appropriate units** in your answers.

(a) What is the average velocity of the train over the first hour of its trip?

Solution:

$$\frac{d(1) - d(0)}{1 - 0} = \frac{(80 \cdot (1)^2 - 50 \cdot (1)^4) - (80 \cdot 0^2 - 50 \cdot 0^4)}{1 - 0} = \frac{80 - 50}{1}$$
$$= 30 \text{ miles per hour}$$

(b) Find a formula for the instantaneous velocity of the train at time t.

Solution:

$$v(t) = d'(t) = 80 \cdot (t^2)' - 50 \cdot (t^4)' = 80 \cdot (2t) - 50 \cdot (4t^3)$$
$$= 160t - 200t^3 \text{ miles/hour}$$

(c) What is the acceleration of the train half an hour after departure?

Solution:

$$a(t) = v'(t) = 160(t)' - 200(t^3)' = 160 \cdot 1 - 200 \cdot (3t^2)$$
$$\Rightarrow a(1/2) = 160 - 200(3/2^2) = 160 - 50 \cdot 3 = 10 \text{ miles/hour}^2$$

(d) Does the train ever go backwards (westward) after starting its trip? If so, when?

Solution: Want to know if/when $v(t) < 0$.
We know $v(t) = 160t - 200t^3 = t(160 - 200t^2)$. Know $t > 0$.
We find that $160 - 200t^2 \geq 0 \Leftrightarrow \frac{4}{5} = \frac{160}{200} \geq t^2$.

The train is going backwards after $t = \dfrac{2}{\sqrt{5}} \approx 0.894427$ hours

3. (12 points) Find the following limits. Your answer must be one of DNE, ∞, $-\infty$ or a number.

(a) $\lim_{x\to 0} \sin\left(\frac{x}{x+1}\right)$

Solution: $= \sin\left(\frac{0}{0+1}\right) = \sin(0) = 0$

(b) $\lim_{x\to 0} \frac{2x+1}{x^3}$

Solution: $\lim_{x\to 0} 2x+1 = 1$ and $\lim_{x\to 0} x^3 = 0$

For small $x > 0$, $\frac{2x+1}{x^3} > 0$ so $\lim_{x\to 0^+} \frac{2x+1}{x^3} = \infty$

For small $x < 0$, $\frac{2x+1}{x^3} < 0$ so $\lim_{x\to 0^-} \frac{2x+1}{x^3} = -\infty$

Therefore $\lim_{x\to 0} \frac{2x+1}{x^3}$ **DNE**

(c) $\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x+3}$

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x+3} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{\sqrt{4x^2+1}\cdot\sqrt{1/x^2}}{(5x+3)\cdot(1/x)} = \lim_{x\to\infty} \frac{\sqrt{4+1/x^2}}{5+3/x} = \frac{\sqrt{4}}{5} = \frac{2}{5}$$

(d) $\lim_{h\to 0} \frac{2e^h-2}{h}$

Solution: $\lim_{h\to 0} \frac{2e^h-2}{h} = f'(0)$ where $f(x) = 2e^x$.

$f'(x) = (2e^x)' = 2e^x \Rightarrow \lim_{h\to 0} \frac{2e^h-2}{h} = f'(0) = 2e^0 = 2$

4. (5 points) Find the equation of the tangent line to the curve $y = x\sin(x)$ at $x = \pi$.

Solution: For $f(x) = x\sin(x)$, we have $f(\pi) = \pi\sin(\pi) = 0$, so the point on the graph is $(\pi, 0)$.

Using the product rule, $y' = (x)' \cdot \sin(x) + x \cdot (\sin(x))' = 1 \cdot \sin(x) + x \cdot \cos(x)$.

Slope of the tangent line $y'(\pi) = \sin(\pi) + \pi\cos(\pi) = 0 + \pi \cdot (-1) = -\pi$.

Point-slop form of the tangent line: $(y - 0) = f'(\pi)(x - \pi) \Rightarrow y = -\pi(x - \pi)$.

5. (6 points) Use the *definition of the limit* to find $f'(1)$ where $f(x) = \sqrt{2x+1}$.
(You may not use any derivative rules.)

Solution: By definition,

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{\sqrt{2(1+h)+1} - \sqrt{2+1}}{h} = \lim_{h\to 0}\frac{\sqrt{2h+3} - \sqrt{3}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{2h+3} - \sqrt{3}}{h} \cdot \frac{\sqrt{2h+3} + \sqrt{3}}{\sqrt{2h+3} + \sqrt{3}} = \lim_{h\to 0}\frac{2h+3-3}{h(\sqrt{2h+3} + \sqrt{3})} = \lim_{h\to 0}\frac{2}{(\sqrt{2h+3} + \sqrt{3})}$$

$$= \frac{2}{(\sqrt{3} + \sqrt{3})} = \frac{1}{\sqrt{3}} \approx 0.57735$$

6. (8 points) Find the x-values of all points on the curve $y = \left(1 + \frac{1}{x}\right) e^x$ at which the tangent line is horizontal.

Solution: For $f(x) = \left(1 + \frac{1}{x}\right) e^x$, using the product rule we find that

$$\begin{aligned} f'(x) &= \left(1 + x^{-1}\right)' e^x + \left(1 + x^{-1}\right) (e^x)' \\ &= \left(0 - x^{-2}\right) e^x + \left(1 + x^{-1}\right) e^x \\ &= \left(1 + x^{-1} - x^{-2}\right) e^x \\ &= \frac{x^2 + x - 1}{x^2} e^x \end{aligned}$$

The line tangent to the curve is horizontal when it has slope zero, i.e. $f'(x) = 0$. We need to find all solutions of $\frac{x^2 + x - 1}{x^2} \cdot e^x = 0$.

Since e^x is always non-zero, we see that $f'(x) = 0$ only when $x^2 + x - 1 = 0$. Using the quadratic formula, we find that this happens for

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \approx -1.61803, 0.61803$$