1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

(a) (4 points) $\lim_{x\to 2} \frac{x^2-4}{2x^2-3x-2}$

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{2x^2-3x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{(2x+1)(x-2)} \\ &= \lim_{x\to 2} \frac{x+2}{2x+1} \\ &= \frac{4}{5}\end{aligned}$$

(b) (4 points) $\lim_{h\to 0} \left(\frac{2}{h^3+2h} - \frac{1}{h}\right)$

$$\begin{aligned}\lim_{h\to 0} \left(\frac{2}{h^3+2h} - \frac{1}{h}\right) &= \lim_{h\to 0} \left(\frac{2-\left(h^2+2\right)}{h^3+2h}\right) \\ &= \lim_{h\to 0} \left(\frac{-h^2}{h^3+2h}\right) \\ &= \lim_{h\to 0} \left(\frac{-h}{h^2+2}\right) \\ &= \frac{0}{2} = 0\end{aligned}$$

(c) (4 points) $\lim_{x\to\infty} \left(\sqrt{4x^2-3x} - 2x\right)$

$$\begin{aligned}\lim_{x\to\infty} \left(\sqrt{4x^2-3x} - 2x\right) &= \lim_{x\to\infty} \left(\sqrt{4x^2-3x} - 2x\right) \cdot \frac{\sqrt{4x^2-3x}+2x}{\sqrt{4x^2-3x}+2x} \\ &= \lim_{x\to\infty} \frac{-3x}{\sqrt{4x^2-3x}+2x} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} \frac{-3}{\sqrt{4-3/x}+2} \\ &= \frac{-3}{\sqrt{4}+2} = -\frac{3}{4}\end{aligned}$$

2. (7 points) Use the limit definition of the derivative on this problem. Find the slope of the tangent line to the curve $y = \dfrac{1}{5-2x}$ at the point $(2,1)$.

$$
\begin{aligned}
m_{\tan} &= \lim_{h\to 0} \frac{\frac{1}{5-2(2+h)} - 1}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{1-(5-2(2+h))}{5-2(2+h)} \right) \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{2h}{5-2(2+h)} \right) \\
&= \lim_{h\to 0} \frac{2}{5-2(2+h)} \\
&= 2
\end{aligned}
$$

3. (7 points) Calculate the equation of the tangent line to $g(x) = \dfrac{1+x}{1+x+x^2}$ at $x=2$.

$$g(2) = \frac{3}{7}$$

$$g'(x) = \frac{(1+x+x^2)\cdot 1 - (1+x)(1+2x)}{(1+x+x^2)^2}$$

$$g'(2) = \frac{7-3\cdot 5}{7^2} = -\frac{8}{49}$$

The equation of the tangent line is $y - \dfrac{3}{7} = -\dfrac{8}{49}(x-2)$

4. (8 points) Let $H(x) = \begin{cases} (x-1)^2 & \text{if } x < 0; \\ e^{x^2} & \text{if } x \geq 0. \end{cases}$

   Is $H(x)$ a continuous function? Use limits to give a careful justification of your answer.

   When $x < 0$, $H(x) = (x-1)^2$. This is a polynomial and is a continuous function.

   When $x > 0$, $H(x) = e^{x^2}$. This is also continuous (the composition of an exponential function and a polynomial).

   Thus we only need check continuity at $x = 0$.

   First note that $H(0) = e^{0^2} = 1$.

$$\begin{aligned} \lim_{x\to 0^+} H(x) &= \lim_{x\to 0^+} e^{x^2} \\ &= e^0 = 1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^-} H(x) &= \lim_{x\to 0^+} (x-1)^2 \\ &= (0-1)^2 = 1 \end{aligned}$$

   Since $\lim_{x\to 0^-} H(x) = \lim_{x\to 0^+} H(x) = H(0)$, $H(x)$ is continuous at $x = 0$.

   Thus $H(x)$ is a continuous function.

5. (8 points) A particle is travelling in a straight line. Its position is given by $x = \left(t^2 - 7\right) e^t$, where x is in feet and t is in seconds. Find all times when the acceleration of the particle is zero.

$$\begin{aligned} \frac{dx}{dt} &= 2te^t + \left(t^2 - 7\right) e^t \\ &= \left(t^2 + 2t - 7\right) e^t \end{aligned}$$

$$\begin{aligned} \frac{d^2x}{dt^2} &= (2t+2)\, e^t + \left(t^2 + 2t - 7\right) e^t \\ &= \left(t^2 + 4t - 5\right) e^t \end{aligned}$$

   We must solve $\left(t^2 + 4t - 5\right) e^t = 0$.

   Since e^t is never zero, we only need to solve

$$\begin{aligned} 0 &= t^2 + 4t - 5 \\ &= (t+5)(t-1) \\ t &= -5, 1 \textit{ seconds} \end{aligned}$$

6. (8 points) Find **two** different points on the curve $y = \dfrac{x}{x-1}$ at which the tangent line passes through the point $(-14, 2)$.

Let (a,b) be such a point.
The tangent line at (a,b) has the form $y - b = m(x-a)$.
Since it passes through $(-14,2)$, we get $2 - b = m(-14-a)$.
We have $b = \dfrac{a}{a-1}$ *because (a,b) is on the curve.*
To compute m, first compute $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x-1)\cdot 1 - x\cdot 1}{(x-1)^2}\\ &= -\frac{1}{(x-1)^2}\\ m &= -\frac{1}{(a-1)^2}\end{aligned}$$

Now eliminate b and m in $2 - b = m(-14-a)$

$$\begin{aligned}2 - \frac{a}{a-1} &= -\frac{1}{(a-1)^2}(-14-a)\\ 2(a-1)^2 - a(a-1) &= -(-14-a)\\ 0 &= a^2 - 4a - 12\\ &= (a-6)(a+2)\\ a &= -2, 6\end{aligned}$$

The points are $\left(-2, \dfrac{2}{3}\right)$ and $\left(6, \dfrac{6}{5}\right)$.