Instructions:

- Your exam contains 5 problems.
- Your exam should contain 6 pages; please make sure you have a complete exam.
- Box in your final answer when appropriate.
- Unless stated otherwise, you **MUST** show work for credit. No credit for answers only. If in doubt, ask for clarification.
- Your work needs to be neat and legible.
- You are allowed one 8.5 × 11 sheet of notes (both sides).

Problem #1 (20 pts)

Problem #2 (15 pts)

Problem #3 (18 pts)

Problem #4 (15 pts)

Problem #5 (12 pts)

TOTAL (80 pts)
1. Calculate the following limits. If the limit does not exist enter DNE. Make sure to show all steps.

(a) (5 points) \[ \lim_{x \to +\infty} \frac{3x^{10} + x^8 + 3}{x^{10} - 3x^6 + 2} = \frac{\beta \cdot \infty}{\infty} = \frac{3 \cdot \infty}{\infty} = 3 \]

(b) (5 points) \[ \lim_{x \to 1} \frac{e^x + 7}{(x-1)^3} = DNE \text{ because} \]

\[
\lim_{x \to 1^+} \frac{e^x + 7}{(x-1)^3} \to \infty \\
\lim_{x \to 1^-} \frac{e^x + 7}{(x-1)^3} \to -\infty 
\]

(c) (5 points) \[ \lim_{x \to 5} \frac{(x^2 - 25) \cdot \sin x}{(x-5) \cdot \cos x} = (x + 5) \frac{\sin x}{\cos x} = 10 \frac{\sin 5}{\cos 5} \]

(d) (5 points) \[ \lim_{x \to 0} \frac{x}{\sqrt{x + 4} - 2} = \frac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2} = \lim_{x \to 0} \frac{x}{x + 4 - 4} = \sqrt{x + 4} + 2 = 4 \]
2. Given the function 
\[ f(x) = \begin{cases} 
    x^2 \sin\left(\frac{1}{x}\right) + 3, & \text{if } x \neq 0 \\
    1, & \text{if } x = 0 
\end{cases} \]

(a) (5 points) Calculate \( \lim_{x \to 0} f(x) \). Make sure to show all steps.

\( -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \) therefore \(- x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \) and

by the squeeze theorem \( \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 \)

So \( \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) + 3 = 3 \)

(b) (5 points) Is \( f \) continuous at 0?

No because \( \lim_{x \to 0} f(x) \neq f(0) \)

(c) (5 points) Find all values of the parameter \( c \), if any, such that the function

\[ g(x) = \begin{cases} 
    x^2 \sin\left(\frac{1}{x}\right) + 3, & \text{if } x \neq 0 \\
    c, & \text{if } x = 0 
\end{cases} \]

is differentiable at 0.

We need to look at \( \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) + 3 - c}{h} \)

We can use the squeeze theorem again to argue that

\( \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \) since from \(-1 \leq \sin\left(\frac{1}{h}\right) \leq 1 \) we get

\(-h \leq h \sin\left(\frac{1}{h}\right) \leq h \) if \( h > 0 \) so \( \lim_{h \to 0^+} h \sin\left(\frac{1}{h}\right) = 0 \) and

\(-h \geq h \sin\left(\frac{1}{h}\right) \geq h \) if \( h < 0 \) so \( \lim_{h \to 0^-} h \sin\left(\frac{1}{h}\right) = 0 \)

So if \( c = 3 \) \( \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = 0 \) and \( g \) is differentiable.

If \( c \neq 3 \) by part a) \( g \) is not continuous at 0

so it cannot be differentiable, so \( \boxed{c = 3} \) is the only value.
3. Below is the graph of a function $y = f(x)$. Compute (no justification necessary): (If a limit does not exist enter DNE)

(a) (3 points) $\lim_{x \to 0} f(x) = 4$

(b) (3 points) $\lim_{x \to 2^-} f(x) = 3$

(c) (3 points) $\lim_{x \to 2^+} f(x) = D \cup E$

(d) (3 points) $\lim_{x \to 2^-} f(x) = 3$

(e) (3 points) $\lim_{x \to 0} \frac{f(x)}{x} = D \cup E$

(f) (3 points) $\lim_{h \to 0} \frac{f(h) - 4}{h} = \int_0^1 (0) = 0$
4. Calculate the derivatives of the following functions:

a) (5 points) \( f(x) = x^{\sqrt{2}} + \sqrt{2} \cdot x^2 + \pi \)

\[
\sqrt{2} \cdot x^{\sqrt{2} - 1} \to 2 \sqrt{2} \cdot x
\]

b) (5 points) \( g(x) = \frac{x \cdot e^x}{x + 2} \)

\[
\frac{(e^x + x \cdot e^x)(x + 2) - x \cdot e^x}{(x + 2)^2} = \frac{2 \cdot e^x}{(x + 2)^2}
\]

\[
e^x \cdot \frac{x^2 + 2x + 2}{(x + 2)^2}
\]

c) (5 points) \( f(x) = \frac{\sqrt{x}}{x} \)

\[
\frac{\frac{1}{2} \cdot e^x - e^x \cdot \sqrt{x}}{(e^x)^2} = \frac{1 - 2 \cdot x}{2 \sqrt{x} \cdot e^x}
\]
5. (15 points) The graph below shows the graph of \( f(x) = x^3 \) and a line \( L \) going through the point \( P = (0, -16) \) and some point \( Q \) on the curve.

(a) (6 points) Find the equation of the tangent line to the graph of \( f(x) \) going through the point \((-2, -8)\).

\[
\begin{align*}
\frac{dx}{dy}(x) &= 3x^2 \\
\frac{dx}{dy}(-2) &= 3 \cdot 4 = 12 \\
y &= -8 + 12(x + 2) \\
\end{align*}
\]

(b) (6 points) The line \( L \) is tangent to the curve at \( Q \). Find the exact equation of the line (an estimate based on the graph will receive no credit).

\[
\begin{align*}
Q &= (x, y) \\
\frac{dx}{dy} &= 3x^2 = \frac{y + 16}{x} \\
3x^2 &= \frac{x^3 + 16}{x} \\
3x^3 &= x^3 + 16 \\
x^3 &= 16 \\
&= 2^3 \\
y &= 2^3 + 12(x - 2) \\
&= 8 + 12(x - 2) \\
&= 12x - 16
\end{align*}
\]