Math 124 (Pezzoli)
Spring 2013
Midterm #1 (100 points)

Instructions:

- Your exam contains 5 problems. The entire exam is worth 70 points. The point value of each problem is clearly marked.
- Your exam should contain 6 pages; please make sure you have a complete exam.
- Box in your final answer when appropriate.
- Unless stated otherwise, you MUST show work for credit. No credit for answers only. If in doubt, ask for clarification.
- Your work needs to be neat and legible.
- You are allowed one 8.5 x 11 sheet of notes (both sides). Graphing calculators are NOT allowed; scientific calculators are allowed. Make sure your calculator is in radian mode.

Problem #1 (15 pts) 

Problem #2 (15 pts) 

Problem #3 (10 pts) 

Problem #4 (15 pts) 

Problem #5 (15 pts) 

TOTAL (100 pts) 

Name ______________________

TA: ______________________

Section: ____________________
1. Calculate the following limits: if the limit exists and it has a finite value, find the value, otherwise decide whether limit does not exist (DNE) or it is $\pm \infty$ or $-\infty$. Make sure to justify all steps.

(a) (5 points) $\lim \limits_{x \to 0} \frac{\sin(2x)}{x} = \lim \limits_{x \to 0} \frac{\sin(2x)}{2x} \cdot 2 = 2$

(b) (5 points) $\lim \limits_{x \to 3^+} \frac{x + e^x}{(3-x)e^x}$

$\lim \limits_{x \to 3^+} \frac{x + e^x}{e^x} = \frac{3 + e^3}{e^3} > 0$

$\lim \limits_{x \to 3^+} \frac{1}{3-x} = -\infty$ Therefore

$\lim \limits_{x \to 3^+} \frac{x + e^x}{(3-x)e^x} = -\infty$

(c) (5 points) $\lim \limits_{t \to 0} \frac{1}{2t \sqrt{1+2t}} - \frac{1}{2t} = \lim \limits_{t \to 0} \frac{1 - \sqrt{1+2t}}{2t \sqrt{1+2t}} - \frac{1 + \sqrt{1+2t}}{2t \sqrt{1+2t}} = \lim \limits_{t \to 0} \frac{1 - (1+2t)}{2t \sqrt{1+2t} (1+\sqrt{1+2t})} = \frac{-1}{\sqrt{1+2t} (1+\sqrt{1+2t})} = \square$
2. Given the function

\[ f(x) = \begin{cases} \frac{\sqrt{x^2 + x^4}}{5x^2 + 3x + 1} & \text{if } x > 0 \\ x & \text{if } x \leq 0 \end{cases} \]

(a) (5 points) Calculate the limit \( \lim_{x \to 0} f(x) \).

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + x^4}}{x^2 (5 + 3 \frac{x}{x^2} + \frac{1}{x^2})} = \frac{x^2 \sqrt{9 + \frac{1}{x^2}}}{x^2 (5 + \frac{3}{x} + \frac{1}{x^2})} = \sqrt{9} = 3
\]

(b) (5 points) Is \( f \) continuous at \( 0 \)? Justify your answer.

\[ f(0) = 0 \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0} x = 0 \]

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sqrt{9x^4 + x^2}}{5x^2 + 3x + 1} = 0 \]

\( f(0) = \lim_{x \to 0} f(x) \) \quad \boxed{\text{YES}} \text{ It is continuous at } 0 \]

(c) (5 points) Is \( f \) differentiable at \( 0 \)? Justify your answer.

\[ \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{\sqrt{9h^4 + h^2}}{5h^2 + 3h + 1} = \frac{h \sqrt{9h^2 + 1}}{h(5h^2 + 3h + 1)} \]

\[ \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sqrt{9h^4 + h^2}}{5h^2 + 3h + 1} \]

\( = 1 \) \quad \boxed{\text{YES}} \text{ it is differentiable at } 0 \]
3. A large ferris wheel is 100 ft in diameter with highest point 110 feet off the ground. It rotates counterclockwise with angular speed \( \omega = \frac{\pi}{10} \) radians/minute. Bob boards the wheel at its lowest position at time \( t = 0 \).

\[
T = \frac{2 \pi}{\omega} = \frac{2 \pi}{\frac{\pi}{10}} = 20 \text{ min}
\]

(a) (5 points) Find parametric equations for Bob's position with respect to the coordinates system shown in the graph above.

\[
x(t) = 50 \cos \left( \frac{\pi}{10} t + \frac{3}{2} \pi \right)
\]
\[
y(t) = 60 + 50 \sin \left( \frac{\pi}{10} t + \frac{3}{2} \pi \right)
\]

(other solutions are possible)

(b) (5 points) At time \( t = 0 \) (the same time Bob starts moving up on the wheel) a bird on the ground 120 feet East of the wheel starts flying up towards a point located directly on top of the wheel 150 feet high (see picture above) at a speed of 25 feet per minute. Who is the first to reach a height of 50 ft? Bob or the bird?

bird's path is line \( y = 150 - 5 \frac{x}{4} \)

set \( y = 50 \) and solve for \( x \)

\( 50 = 150 - 5 \frac{x}{4} \)

\( x = 80 \). At height 50 feet the bird is \( \frac{4}{5} \)

Bird has located at \( P(80, 50) \) and it has travelled a distance \( d = \sqrt{(120-80)^2 + 50^2} = 10 \sqrt{41} \)

in time \( t = \frac{d}{v} = 10 \sqrt{41} \approx 2.56 \text{ min} \). Since \( T = 20 \) it takes Bob 25 s min to reach a height of 60 feet and at time \( t = 10 \sqrt{41} / 25 \) Bob's height is only \( 60 + 50 \sin \left( \frac{\pi}{10} \cdot \frac{10 \sqrt{41}}{25} + \frac{3}{2} \pi \right) \approx 25.33 \)
4. Calculate the derivatives of the following functions, you do not need to simplify your result:

a) (5 points) \( f(x) = e^x \cdot x^{\sqrt{2}} + \pi^2 \)

\[
\frac{d}{dx}(x) = e^x \cdot x^{\sqrt{2}} + \sqrt{2} \cdot e^x \cdot x^{\sqrt{2}-1}
\]

b) (5 points) \( g(x) = \frac{3 \sin(x)}{\sqrt{x}} \)

\[
\frac{3 \cos(x) \cdot \sqrt{x} - \frac{3 \sin(x)}{2 \cdot \sqrt{x}}}{x}
\]

c) (5 points) \( h(x) = \frac{e^x \cdot \cos(x)}{x} \)

\[
\frac{(e^x \cos(x) - e^x \sin(x)) \cdot x - e^x \cos(x)}{x^2}
\]
5. Given \( f(x) = -x^2 - 4x - 1 \).

(a) (7 points) Find the equation(s) of all tangent line(s) to the graph of \( f \) through \( P = (0, -1) \)

\[ f(0) = -1 \quad P \text{ is on the curve} \]
\[ f'(x) = -2x - 4 \quad f'(0) = -4 \]

Tangent is \( y + 1 = -4x \)

(b) (8 points) Find the equation(s) of all tangent line(s) to the graph of \( f \) through \( Q = (0, 0) \)

\( Q \) is not on the curve. Point of tangency is \( R = (x, y) \). Slope of tangent is 1) \( m = -2x - 4 \) 2) \( m = \frac{y - 0}{x - 0} \)

Set them equal and replace \( y \) with \( f(x) \):

\[-2x - 4 = \frac{-x^2 - 4x - 1}{x} \quad \text{solve for } x \]

\[-2x^2 - 4x = -x^2 - 4x - 1 \quad x^2 = 1 \]

\[x = \pm 1 \quad \text{either } R = (1, f(1)) = (1, -6) \]
\[m = f'(1) = -6 \quad \text{tangent is } y + 6 = -6(x - 1) \]

or \( R = (-1, f(-1)) = (-1, 6) \)
\[m = f'(-1) = -2 \quad \text{tangent is } y - 2 = -2(x + 1) \]