Solutions to Autumn 2025 Math 124 Final Exam

1. (a) $\lim_{x\to\infty} x\ln(x+2) - x\ln(x) = \lim_{x\to\infty} x\ln\left(\frac{x+2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+2}{x}\right)}{1/x}$

   $=^{\mathrm{LH}} \lim_{x\to\infty} \frac{\frac{-2/x^2}{\frac{x+2}{x}}}{-1/x^2} = \lim_{x\to\infty} \frac{2x}{x+2} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$

   (b) From $y = x^x$ get $\ln y = \ln x^x = x\ln x$. Then,

   $$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} =^{\mathrm{LH}} \lim_{x\to 0} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+}(-x) = 0$$

   So, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$.

   (c) $\lim_{x\to\frac{\pi}{2}} \frac{\sin(3x)+1+3\cos(x)}{\cos(5x)+1-\sin(x)} =^{\mathrm{LH}} \lim_{x\to\frac{\pi}{2}} \frac{3\cos(3x)-3\sin(x)}{-5\sin(5x)-\cos(x)} = \frac{3}{5}$

2. (a) $f'(x) = (-x)\cdot e^{-\frac{x^2}{2}} + (2x)\cdot\sin(x) + (\cos x)\cdot(x^2)$

   (b) $g'(x) = \sin(x^3)^{\ln(x)} \cdot \left(\left(\frac{1}{\sin(x^3)}\cdot\cos(x^3)\cdot 3x^2\right)\cdot(\ln(x)) + \big(\ln(\sin(x^3))\cdot\left(\frac{1}{x}\right)\big)\right)$

   (c) $h'(x) = \frac{\left(\frac{x}{1+x^2}\right) - \arctan(x)}{x^2}$

3. (a) $\frac{dy}{dx} = \frac{1}{3}\cdot x^{-2/3}$ and $L(x) = 2 + \frac{1}{12}(x-8)$

   (b) $(8.12)^{1/3} \approx L(8.12) = 2 + \frac{1}{12}(0.12) = 2.01$

4. Differentiate both sides to get

   $$e^y + xe^y y' + y'\sin(x) + y\cos x - 2yy' = 0$$

   when $x = 0$ and $y = 1$ you get $e + 1 - 2y' = 0$ so the slope is $\frac{e+1}{2}$ and the equation of the tangent line is

   $$y - 1 = \frac{e+1}{2}x.$$

5. (a) The tangent is horizontal when

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$$

   so when $dy/dt = 0$. Solve $3\cos 3t = 0$ to get $3t = \frac{\pi}{2} + k\pi$. Since $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $t = \pm\frac{\pi}{6}$. So the two points are $(0.5, \pm 1)$.

   (b) $(\cos 2t, \sin 2t) = (-1/2, 0)$ when $t = \pm\pi/3$ where the slopes are given by

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{-2\sin 2t}$$

   so

   $$\frac{3\cos\pi}{-2\sin(2\pi/3)} = \sqrt{3} \text{ and } \frac{3\cos(-\pi)}{-2\sin(-2\pi/3)} = -\sqrt{3}$$

   so the two tangent lines are $y = \sqrt{3}x \pm \frac{\sqrt{3}}{2}$.

6. If y is the length of OP, then

$$\theta = \arctan\left(\frac{8}{y}\right) + \arctan\left(\frac{5}{y}\right).$$

Differentiate with respect to t to get

$$\frac{d\theta}{dt} = \left(\frac{1}{1+\left(\frac{8}{y}\right)^2}\cdot\left(\frac{-8}{y^2}\right) + \frac{1}{1+\left(\frac{5}{y}\right)^2}\cdot\left(\frac{-5}{y^2}\right)\right)\frac{dy}{dt} = \left(\frac{-8}{y^2+64} + \frac{-5}{y^2+25}\right)\frac{dy}{dt}$$

So when $y = 10$ we get

$$\frac{d\theta}{dt} = \left(\frac{-8}{164} + \frac{-5}{125}\right)\cdot 6 = -\frac{546}{1025}$$

7. If the radius of the cylinder is r and its height is h, then

$$\frac{r}{8} = \frac{6-h}{6}$$

so

$$r = \frac{4}{3}(6-h) \qquad \text{OR} \qquad h = 6 - \frac{3r}{4}$$

the volume of the cone $V = \pi r^2 h$ is

$$V = \pi\left(\frac{4}{3}(6-h)\right)^2 h = \frac{16\pi}{6}(36h - 12h^2 + h^3) \qquad \text{OR} \qquad V = \pi r^2\left(6 - \frac{3r}{4}\right) = \frac{3\pi}{4}(8r^2 - r^3)$$

so

$$V' = \frac{16\pi}{6}(36 - 24h^2 + 3h^2) = \frac{16\pi}{2}(h-2)(h-6) \qquad \text{OR} \qquad V = \frac{3\pi}{4}(16r - 3r^2) = \frac{3\pi}{4}(16-3r)r$$

So the critical number is

$$h = 2 \qquad \text{OR} \qquad r = \frac{16}{3}.$$

(Note that $0 \le h \le 6$ and $0 \le r \le 8$ so the other roots of V' are endpoints.)

To verify max you can do one of:

- Check function values

$$V(0) = 0, V(2) = \frac{512\pi}{9}, V(6) = 0 \qquad \text{OR} \qquad V(0) = 0, V\left(\frac{16}{3}\right) = \frac{512\pi}{9}, V(8) = 0$$

- Check the sign change for V' at the critical point

$$V'(1) = \frac{80\pi}{3} > 0, V'(3) = -16\pi < 0 \qquad \text{OR} \qquad V'(5) = \frac{15\pi}{4} > 0, V'(6) = -9\pi < 0$$

- Check the sign of $V'' = \frac{16\pi}{3}(-8+2h)$ or $V'' = \frac{3\pi}{4}(16-6r)$ at the critical point

$$V''(2) = \frac{16\pi}{3}(-4) < 0 \qquad \text{OR} \qquad V''(16/3) = \frac{3\pi}{4}(16-32) < 0.$$

So the dimensions are $h = 2$ and $r = 16/3$.

8. (a) The number $x = 0$ is not in the domain so there is no $y-$ intercept. $f(x) = 0$ when $3x^2 = 1$ so $x = \pm 1/\sqrt{3}$ are the $x-$ intercepts/

(b) $x = 0$ is the vertical asymptote. From

$$\lim_{x\to\infty} \frac{3x^2 - 1}{3x^3} =^{\text{LH}} \lim_{x\to\infty} \frac{6x}{9x^2} = \lim_{x\to\infty} \frac{2}{3x} = 0$$

and

$$\lim_{x\to-\infty} \frac{3x^2 - 1}{3x^3} =^{\text{LH}} \lim_{x\to-\infty} \frac{6x}{9x^2} = \lim_{x\to-\infty} \frac{2}{3x} = 0$$

$y = 0$ is the horizontal asymptote (on both sides).

(c) From

$$f'(x) = \frac{(1-x)(1+x)}{x^4}$$

f is increasing on $(-1, 0)$ and $(0, 1)$ and decreasing on $(\infty, -1)$ and $(1, \infty)$. There is a local minimum at $(-1, -2/3)$ and a local maximum at $(1, 2/3)$.

(d) From

$$f''(x) = \frac{2(x - \sqrt{2})(x + \sqrt{2})}{x^5}$$

the graph is concave up on $(-\sqrt{2}, 0)$ and $(0, \sqrt{2}$ and concave down on $(\infty, \sqrt{2})$ and $(0, \sqrt{2})$. The inflection points are $(-\sqrt{2}, -5\sqrt{2}/12)$ and $(\sqrt{2}, 5\sqrt{2}/12)$

(e) Note that the function is odd so the graph is symmetric with respect to the origin.