Solutions to Math 124 Winter 2024 Final

1. (a) $\lim_{t\to 2} \frac{3^t - 9}{t-2} =^{\text{LH}} \lim_{t\to 2} \frac{\ln 3 \cdot 3^t}{1} = 9\ln 3$

   (b) $\lim_{x\to\infty} \frac{\sqrt{x^2+x+1}}{\sqrt{7x^2+3}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}{\sqrt{7+\frac{3}{x^2}}} = \frac{1}{\sqrt{7}}$

   (c) $\lim_{x\to 0} \frac{\sin(5x)}{\tan(3x)} =^{\text{LH}} \lim_{x\to 0} \frac{5\cos(5x)}{3\sec^2(3x)} = \frac{5}{3}$

2. (a) $f'(x) = \arctan(x^2+1) + x \cdot \frac{2x}{1+(x^2+1)^2}$

   (b) $g'(x) = \frac{e^{e^x} \cdot e^x \cdot \ln x - \frac{e^{e^x}}{x}}{(\ln x)^2}$

   (c)

$$y = x^{\sqrt{x}+1}$$

$$\ln y = \ln\left(x^{\sqrt{x}+1}\right) = (\sqrt{x}+1)\ln x$$

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + (\sqrt{x}+1)\cdot\frac{1}{x}$$

$$y' = \left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}+1}{x}\right)x^{\sqrt{x}+1}$$

3. (a) $f'(2) = 1$

   (b) $f''(0) = -1$

   (c) $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = f'(0) = 0$

   (d) $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = f'(2) = 1$

   (e) $\lim_{x\to 3^+} \frac{f'(x)}{x-3} = \lim_{x\to 3^+} \frac{f'(x)-f'(3)}{x-3} = 2$

   (f) $(-2,0)\cup(1,4)$

   (g) $x = 0, 1, 3$

   (h) $(1,2)\cup(3,4)$

   (i) $x = 1, 3$

4. Implicit Differentiation:

$$3(p-20)^2\frac{dp}{dq} + \frac{dp}{dq}q + p + 3q^2 = 0.$$

   When $q = 16$, $p = 5$

$$3(-15)^2\frac{dp}{dq} + \frac{dp}{dq}\cdot 16 + p + 3\cdot 16^2 = 0.$$

   so $\frac{dp}{dq} = -\frac{773}{691}$ and the tangent line is

$$p - 5 = -\frac{773}{691}(q-16)$$

   so the approximation is

$$p - 5 \approx -\frac{773}{691}(17-16)$$

   so $p \approx 5 - \frac{773}{691} \approx 3.88$

5. (a) Solving

$$0 = y = (t-2)^3 - 3(t-2) = (t-2)(t^2 - 4t + 1)$$

we have $t = 2$ or $t \pm \sqrt{3}$.

(b)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t-2)^2 - 3}{2(t-2)}$$

When $t = 2 - \sqrt{3}$,

$$\frac{dy}{dx} = \frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = -\sqrt{3}$$

so the tangent line is at the point $x = (2-\sqrt{3}-2)^2 = 3$ and $y = (2-\sqrt{3}-2)^3 - 3(2-\sqrt{3}-2) = 0$ is

$$y = -\sqrt{3}(x-3)$$

(c) $y = \sqrt{3}(x-3)$.

6. Given $\frac{dV}{dt} = 5 - 3 = 2$, we want $\frac{dh}{dt}$. If r is the radius of the surface of the water, from similar triangles

$$\frac{2}{6} = \frac{r}{h}$$

or $r = h/3$. So the volume of water is

$$V = \frac{\pi r^2 h}{3} = \frac{\pi (h/3)^2 h}{3} = \frac{\pi h^3}{27}$$

and

$$\frac{dV}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$$

so $\frac{dh}{dt} = \frac{2}{\pi}$ meters per minute.

7. Let x be the distance from the first pole to where the wire is connected to the ground. Then, the length of the wire is

$$f(x) = \sqrt{20^2 + x^2} + \sqrt{(60-x)^2 + 40^2}$$

and

$$f'(x) = \frac{2x}{2\sqrt{20^2 + x^2}} + \frac{-2(60-x)}{2\sqrt{(60-x)^2 + 40^2}} = 0$$

when

$$x\sqrt{(60-x)^2 + 40^2} = (60-x)\sqrt{20^2 + x^2}$$

or

$$x^2\left((60-x)^2 + 40^2\right) = (60-x)^2(20^2 + x^2)$$

so $x = 20$ (or $x = -60$ which does not make sense). Checking endpoints and the critical point

$$f(0) \approx 92.1,\ f(20) \approx 84.9,\ f(60) \approx 103.2$$

so the minimum amount of wire is approximately 84.9 meters.(You can also verify the sign change of f' at $x = 20$ to show it must give the minimum length of wire.)

8. (a) When $y = 0$, the $x-$intercept is at $x = 1$. The $y-$intercept is at $y = f(0) = 1$.

(b) The domain is all numbers so there is no vertical asymptote.

$$\lim_{x \to \infty} e^x (x-1)^2 = \infty$$

and

$$\lim_{x \to -\infty} e^x (x-1)^2 = 0$$

so $y = 0$ is the horizontal asymptote.

(c)

$$f'(x) = e^x (x-1)^2 + 2e^x (x-1) = e^x (x^2 - 1)$$

so the critical numbers are $x = \pm 1$. The function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$ so it has a local max at $x = -1$ and a local min at $x = 1$.

(d)

$$f''(x) = e^x (x^2 - 1) + e^x (2x) = e^x (x^2 + 2x - 1) = 0$$

when $x = -1 \pm \sqrt{2}$ which give points of inflection because f is concave up ($f'' < 0$) on $(-\infty, -1-\sqrt{2})$ and $(-1+\sqrt{2})$ and concave down ($f'' < 0$) on $(-1-\sqrt{2}, -1+\sqrt{2})$.

(e) Graph of $y = f(x)$