Solutions to Math 124 Winter 2024 Final

1. (a) $\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)=\lim_{x\to 1}(\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}=\lim_{x\to 1}\frac{x-1}{(x-2)(x-1)}=\lim_{x\to 1}\frac{1}{x-2}=$ -1

   (b) $\lim_{y\to 4^+}\frac{\sqrt{y}}{(y-4)^6}=\infty$ and $\lim_{y\to 4^-}\frac{\sqrt{y}}{(y-4)^6}=\infty$ so $\lim_{y\to 4}\frac{\sqrt{y}}{(y-4)^6}=\infty$

   (c) $\lim_{x\to 1}\frac{\ln x}{\tan(\pi x)}=^{\text{LH}}\lim_{x\to 1}\frac{\frac{1}{x}}{\pi\sec^2(\pi x)}=\frac{1}{\pi}$

2. (a) $f'(x)=\cos(\sqrt{3x})\cdot\frac{\sqrt{3}}{2\sqrt{x}}+\frac{1}{2\sqrt{\sin(3x)}}\cdot 3\cos(3x).$

   (b) $f'(x)=\dfrac{\left(\frac{3\arcsin(7x)}{1+3x}+\frac{7\ln(1+3x)}{\sqrt{1-49x^2}}\right)(e^{2x}+x)+\ln(1+3x)\left(\arcsin(7x)\right)(2e^{2x}+1)}{(e^{2x}+x)^2}$

   (c)

$$\ln(g(t))=\ln\left((\cos(at))^{\sqrt{bt+c}}\right)=\sqrt{bt+c}\cdot\ln\left(\cos(at)\right)$$

$$\frac{g'(t)}{g(t)}=\frac{b\cdot\ln\left(\cos(at)\right)}{2\sqrt{bt+c}}+\sqrt{bt+c}\cdot\frac{-a\sin(at)}{\cos(at)}$$

$$g'(t)=\left(\frac{b\cdot\ln\left(\cos(at)\right)}{2\sqrt{bt+c}}-a\tan(at)\sqrt{bt+c}\right)\cdot\left(\cos(at)\right)^{\sqrt{bt+c}}$$

3. (a) $\lim_{x\to -1}f'(x)$ DNE

   (b) $f'(0)=4$

   (c) $\lim_{x\to 1^-}f''(x)=-\infty$

   (d) $\lim_{h\to 0}\frac{f(h)-f(0)}{h}=4$

   (e) $\lim_{x\to 3}\frac{f'(x)+2}{x-3}=0$

   (f) $g'(-2)=f'(f'(-2))\cdot f''(-2)=-8$

   (g) $x=-1,0,3$

   (h) $x=-2,1.6$

   (i) $x=-1,4.4$

4. (a) First: $8^{1/3}+1^{1/3}=4+1=5$. Plug in $x=8$ and $y=1$ in

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

   to get $y'=-1/2$ so the tangent line is $y-1=-\frac{1}{2}(x-8)$.

   (b) Plug in $x=8$, $y=1$, $y'=-1/2$ in

$$-\frac{2}{9}x^{-4/3}-\frac{2}{9}y^{-4/3}y'y'+\frac{2}{3}y^{-1/3}y''=0$$

   to get $y''=5/48$ so the graph is concave up.

5. (a) From $t_1^2+1=y=t_2^2+1$ we get $t_1=\pm t_2$ since we want them distinct, $t_1=-t_2$. From $t_1^3-t_1+1=x=t_2^3-t_2+1=(-t_1)^3+t_1+1$ we get $t_1^3-t_1=-t_1^3+t_1$ so $0=t_1^3-t_1=t_1(t_1^2-1)$ so $t_1=\pm 1$. The point of self-intersection is

$$(x(1),y(1))=(x(-1),y(-1))=(1,2).$$

(b) The derivative is

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t}{3t^2-1}$$

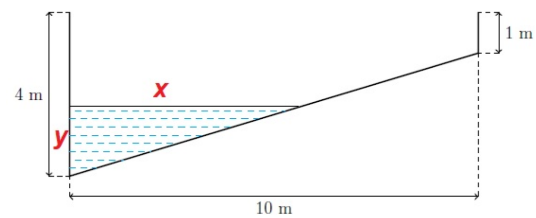
At $t=1$, the slope is 1 and the tangent line is $y-2=x-1$.
At $t=-1$, the slope is -1 and the tangent line is $y-2=-(x-1)$.

(c) When $0=\frac{dx}{dt}=3t^2-1$ we have $t=\pm 1/\sqrt{3}$:

$$\left(x\left(\frac{1}{\sqrt{3}}\right),y\left(\frac{1}{\sqrt{3}}\right)\right)=\left(-\frac{2\sqrt{3}}{9}+1,\frac{4}{3}\right)\text{ and }\left(x\left(-\frac{1}{\sqrt{3}}\right),y\left(-\frac{1}{\sqrt{3}}\right)\right)=\left(\frac{2\sqrt{3}}{9}+1,\frac{4}{3}\right)$$

6. Given $\frac{dV}{dt}=-0.25$ we want $\frac{dy}{dt}$. The volume is $V=5\cdot\frac{xy}{2}$.
Using similar triangles

$$\frac{10}{3}=\frac{x}{y}$$



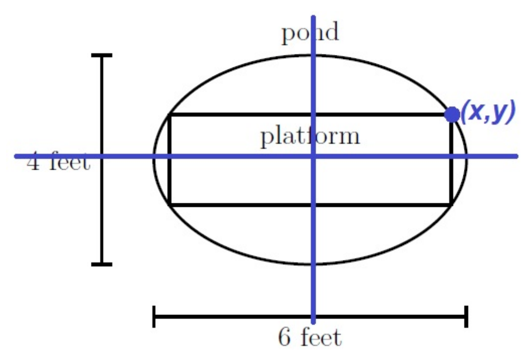
so $x=10y/3$ and $V=25y^2/3$. So

$$\frac{dV}{dt}=\frac{50y}{3}\cdot\frac{dy}{dt}.$$

When $y=2$ we get $\frac{dy}{dt}=-0.0075$ meters per minute.

7. With the center at the origin and the righmost point of the platform at (x,y), the area is $A=4xy$ with the constraint

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$



so we want to maximize

$$A(x)=4x\cdot 2\sqrt{1-\frac{x^2}{9}}.$$

From

$$A'(x)=8\sqrt{1-\frac{x^2}{9}}+8x\cdot\frac{-2x/9}{\sqrt{1-\frac{x^2}{9}}}=0$$

we get $x=3\sqrt{2}$ and $A=12$ squared feet.

8. (a) The only candidate for the vertical asymptote is $x=0$. We check:

$$\lim_{x\to 0}\frac{x^2+x+1}{x^2}=\infty.$$

(b) Checking the limits:

$$\lim_{x\to\infty} \frac{x^2+x+1}{x^2} = 1$$

and

$$\lim_{x\to-\infty} \frac{x^2+x+1}{x^2} = 1.$$

so $y = 1$ is the horizontal asymptote (on both sides).

(c) From

$$f'(x) = \frac{(2x+1)\cdot x^2 - (x^2+x+1)\cdot 2x}{x^4} = -\frac{x+2}{x^3}$$

the only critical number is $x = -2$. Note that $x = 0$ is not a critical number because it is not in the domain of the function.

(d) Decreasing on $(-\infty, -2)$ and $(0, \infty)$. Increasing on $(-2, 0)$. At $(-2, 3/4)$ it has a local minimum.

(e) From

$$f''(x) = -\frac{x^3 - (x+2)\cdot 3x^2}{x^6} = \frac{2x+6}{x^4} = 0$$

we get $x = -3$. The graph is concave up on $(-3, 0)$ and $(0, \infty)$, concave down on $(-\infty, -3)$ and has an inflection point at $(-3, 7/9)$.