Solutions to Winter 2023 Math 124 Final Exam

1. (a) $\lim_{x\to 0} \dfrac{x\sin x}{1-\cos x} =^{\text{LH}}= \lim_{x\to 0} \dfrac{\sin x + x\cos x}{\sin x} =^{\text{LH}}= \lim_{x\to 0} \dfrac{\cos x + \cos x - x\sin x}{\cos x} = 2.$

   (b) $\lim_{x\to 1} \dfrac{x^2-3x+2}{x-1} = \lim_{x\to 1} \dfrac{(x-2)(x-2)}{x-1} = \lim_{x\to 1}(x-2) = -1$

   (c) First,

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} =^{\text{LH}} \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$$

   so $\lim_{x\to\infty} \cos\left(\dfrac{\ln(\ln x)}{\ln x}\right) = \cos 0 = 1$

2. (a) $y' = 20(1+\sqrt{t})^{19} \cdot \dfrac{1}{2\sqrt{t}} \cdot (1+t)^{23} + (1+\sqrt{t})^{20} \cdot 23(1+t)^{22}$

   (b) $y' = \dfrac{1}{3}(\sin x + \tan^3 x)^{-\frac{2}{3}}(\cos x + 3\tan^2 x \sec^2 x)$

   (c) Logarithmic differentiation:

$$\ln y = \ln\left((1+\cos x + e^x)^{\sin x}\right) = \sin x \cdot \ln(1+\cos x + e^x)$$

$$\frac{y'}{y} = \cos x \cdot \ln(1+\cos x+e^x) + \sin x \cdot \frac{-\sin x + e^x}{1+\cos x + e^x}$$

   So,

$$y' = \left(\cos x \cdot \ln(1+\cos x + e^x) + \frac{\sin x(-\sin x + e^x)}{1+\cos x+e^x}\right) \cdot (1+\cos x + e^x)^{\sin x}$$

3. (a) $\lim_{x\to -3^+} f'(x) = 15$, (b) $f''(-7) = -\dfrac{10}{3}$, (c) $-6, 2$, (d) $(5,7)$, (e) $5, 7$ (f) $f(-0.5) = 22.5$

4. (a) From

$$y' = \frac{4y-3x^3}{2y-4x}$$

   the tangent is vertical when $y = 2x$, so we have

$$x^3 - 4x(2x) + (2x)^2 = x^3 - 4x^2 = 0$$

   so $x = 4$ (notr$x = 0$ is not possible from picture) and $y = 2x = 8$.

   (b) The tangent line approximation is $y - 3 \approx \frac{5}{2}(x-3)$ so $x \approx 2.98$.

5. (a) $x'(t) = \cos(\pi t) - \pi t \sin(\pi t)$

   (b) $y'(t) = \sin(\pi t) + \pi t\cos(\pi t)$

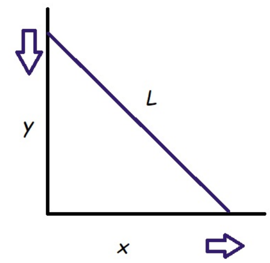
   (c) When $0 = x(t) = t\cos(\pi t)$ we have $t = 0$ (which gives the origin), $t = /2$ or $t = 3/2$. Since $y(3/2) = -3/2 < 0$ that is the t we need. The slope is

$$\left.\frac{dy}{dx}\right|_{t=3/2} = \left.\frac{\sin(\pi t) + \pi t\cos(\pi t)}{\cos(\pi t) - \pi t \sin(\pi t)}\right|_{t=3/2} = -\frac{2}{3\pi}$$

   so the tangent line is $y = -\dfrac{2}{3\pi}x - \dfrac{3}{2}$.

   (d) $\lim_{t\to\infty} \sqrt{(\cos(\pi t) - \pi t\sin(\pi t))^2 + (\sin(\pi t) + \pi t \cos(\pi t))^2} = \lim_{t\to\infty} \sqrt{1+\pi^2 t^2} = \infty$

6. From $x^2 + y^2 = L^2$ we get



$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

so when $x = 5$, $dx/dt = 0.6$, and $dy/dt = -0.25$ we get $y = 12$. So $L = \sqrt{x^2 + y^2} = 13$ meters.

7. The volume is constant

$$\frac{4}{3}\pi r^3 + a^3 = 1000$$

so $a = \left(1000 - \frac{4}{3}\pi r^3\right)^{1/3}$. The total surface area is

$$A = 4\pi r^2 + 6a^2 = 4\pi r^2 + 6\left(1000 - \frac{4}{3}\pi r^3\right)^{2/3}$$

where $0 \leq r \leq \left(\frac{750}{\pi}\right)^{1/3}$ so

$$A' = 8\pi r + 6 \cdot \frac{2}{3}\left(1000 - \frac{4}{3}\pi r^3\right)^{-1/3}(-4\pi r^2) = 0$$

gives $r = 0$ or $\left(1000 - \frac{4}{3}\pi r^3\right)^{1/3} = 2r$ so the critical number is $r = \left(\frac{750}{6+\pi}\right)^{1/3} \approx 4.345$. Checking values at the critical number and the endpoints

$$S(0) = 600, \quad S\left(\left(\frac{750}{\pi}\right)^{1/3}\right) \approx 483.598, \quad S\left(\left(\frac{750}{6+\pi}\right)^{1/3}\right) \approx 690.412$$

So the minimum happens when $r = \left(\frac{750}{\pi}\right)^{1/3} \approx 4.345$ and $a = 0$ so we only have a sphere and the maximum happens when $r = \left(\frac{750}{6+\pi}\right)^{1/3} \approx 4.345$ and $a = \left(\frac{6000}{6+\pi}\right)^{1/3} \approx 8.690$.

8. (a) We have $f'(x) = \frac{6}{x^2} - \frac{12}{x^3} = 0$ when $x = 2$ and is undefined when $x = 0$. The function is decreasing on $(0, 2)$.

   (b) We have $f'(x) = -\frac{12}{x^3} + \frac{36}{x^4} = 0$ when $x = 3$ and is undefined when $x = 0$. The function is concave up on $(0, 3)$.

   (c) The limits are 2, 2, ∞, and ∞.