Answers to Math 124 Spring 2024 Final

1. (a) $\lim_{x\to-\infty} \dfrac{2x^3-4x^2+6x+8}{8x^3-6x^2+8x+1} = \lim_{x\to-\infty} \dfrac{x^3\left(2-\frac{4}{x}+\frac{6}{x^2}+\frac{8}{x^3}\right)}{x^3\left(8-\frac{6}{x}+\frac{8}{x^2}+\frac{1}{x^3}\right)} = \lim_{x\to-\infty} \dfrac{2-\frac{4}{x}+\frac{6}{x^2}+\frac{8}{x^3}}{8-\frac{6}{x}+\frac{8}{x^2}+\frac{1}{x^3}} = \dfrac{2}{8} = \dfrac{1}{4}$

   (b) $\lim_{x\to1} \dfrac{x^2-1}{\sqrt{x^2+3}-2} \cdot \dfrac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \lim_{x\to1} \dfrac{(x^2-1)\left(\sqrt{x^2+3}+2\right)}{x^2-1} = \lim_{x\to1} \sqrt{x^2+3}+2 = 4$

   (c) $\lim_{x\to0^+} \arctan(\ln(x)) = -\dfrac{\pi}{2}$

   (d) $\lim_{x\to1} \left(\dfrac{e^{x-1}-1}{x-1}\right) =^{\text{LH}} \lim_{x\to1} \left(\dfrac{e^{x-1}}{1}\right) = 1$

2. (a) $f'(x) = \dfrac{e^x - (xe^x+e^x)\ln(3x)}{(xe^x)^2}$

   (b) $g'(x) = \dfrac{\left(2x+\frac{1}{2\sqrt{x}}\right)\cos(x^2+\sqrt{x})}{2\sqrt{\sin(x^2+\sqrt{x})}}$

   (c) $h'(x) = (2\ln x+2)x^{2x} + 2x + \ln 2 \cdot 2^x$

   (d) $k'(x) = \dfrac{8x}{1+16x^4}$

3. (a) $f'(-5) = -1$

   (b) $f'(-1) = -1$

   (c) $\lim_{x\to-3} f'(x) = 1$

   (d) $\lim_{x\to\infty} f(x) = -\infty$

   (e) $\lim_{x\to\infty} f'(x) = -1$

   (f) 1

   (g) $-4.5, -1.5, 1, 2$

   (h) $-1.5, 2$

   (i) 5

   (j) $(-\infty, -3)$, $(0, 1)$

4. Given $\dfrac{da}{dt} = 15$ and $\dfrac{db}{dt} = -5$, we want $\dfrac{d\theta}{dt}$ where the relation is $\tan\theta = b/a$. So

$$\sec^2\theta \frac{d\theta}{dt} = \frac{a\frac{db}{dt} - b\frac{da}{dt}}{a^2}$$

   and with the given information the answer is $-2/17$ radians per minute.

5. Minimize

$$M = 4ah + a^2$$

   under the constraint

$$a^2h = 4$$

   to get $a = 2$ and $h = 1$. Do a first derivative test on $M'(a)$ to verify it is a minimum.

6. (a) From

$$\frac{dy}{dx} = \frac{2\cos(2t)}{-\sin t} = 0$$

you get $t = \pm\frac{\pi}{4} + \pi k$ (k integer) and the four points $\left(\pm\frac{\sqrt{2}}{2}, \pm 1\right)$.

(b) At $t = \pi/3$, $\frac{dy}{dx} = 2/\sqrt{3} > 0$ so the function is increasing. Also,

$$\frac{d^2y}{dx^2} = -\frac{4\sin(2t)\sin(t) + 2\cos(2t)\cos(t)}{\sin^3 t}$$

so at $t = \pi/3$ its value is $-5\sqrt{3} < 0$ so the graph is concave down. The correct picture is B.

7. (a) From

$$\frac{1}{y}y' + y' + y^2 + 2xyy' = 0$$

we get $y' = -0.1$ at $(4, 1)$ so the tangent line is $y = -0.1(x-4)+1$ and the approximation gives $y \approx 0.99$.

(b) From

$$-y^{-2}y'y' + \frac{1}{y}y'' + y'' + 2yy' + 2yy' + 2xy'y' + 2xyy'' = 0$$

we get $y'' = 33/1000 > 0$ so the graph is concave up and the answer is an underestimate.

8. (a) $f(0) = 0$ so $(0, 0)$.

(b) No vertical asympototes. From

$$\lim_{x\to\pm\infty} \frac{x}{x^2+1} = 0$$

$y = 0$ is the horizontal asymptote (on both sides.)

(c) From

$$f'(x) = \frac{-x^2+1}{(x^2+1)^2} = 0$$

get $x = \pm 1$.

(d) Increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$ and $(1, \infty)$. Recall that the function is: $f(x) = \dfrac{x}{x^2+1}$.

(e) Local maximum at $(1, 1/2)$ and local minumum at $(-1, -1/2)$

(f) From

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0$$

get $x = 0$ and $x = \pm\sqrt{3}$. Check the sign of f'' to get that graph is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

(g) $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$, $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

(h) .