Answers to Math 124 Spring 2023 Final Exam

1. (a)

$$\lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\frac{\ln x}{x^{-1}} =^{\text{LH}} \lim_{x\to 0^+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0^+}(-x) = 0$$

and

$$\lim_{x\to 0^+}\left(\frac{1}{x}+\frac{1}{\ln x}\right) = \infty$$

(b) Since $9-(f(2))^2 = 0$ we have $f(2) = \pm 3$. Using LH,

$$\lim_{x\to 2}\frac{x^2-4}{9-(f(x))^2} =^{\text{LH}} \lim_{x\to 2}\frac{2x}{-2f(x)f'(x)} = \frac{4}{-2f(2)f'(2)} = 5$$

we get $f'(2) = \pm 2/15$.

(c) False/False/False/True/False

2. (a) $h'(x) = \dfrac{4\sin(x)\cos(x)}{2\sqrt{2\sin^2(x)+1}}$.

(b) $g'(x) = (2ax)e^{-cx} - c(ax^2+b)e^{-cx}$

(c) Logarithmic differentiation:

$$\ln y = x^2\cdot\ln(\cos x)$$
$$\frac{y'}{y} = 2x\ln(\cos x) + \frac{x^2(-\sin x)}{\cos x}$$
$$y' = \left(2x\ln(\cos x) - \frac{x^2\sin x}{\cos x}\right)(\cos x)^{x^2}$$

3. (a) $\displaystyle\lim_{x\to 0}\frac{f(x)}{x} = 2$

(b) $f''(11) = \dfrac{2}{3}$

(c) -3 and 12

(d) $(3,9)$ and $(15,\infty)$

(e) $g'(x) = f'(f(x))\cdot f'(x)$ so $g'(0) = f'(f(0))\cdot f'(0) = f'(0)\cdot f'(0) = 4$

4. (a) Differentiate $4x^3+3y+3xy'+4y^3y' = 0$ and solve

$$y' = -\frac{4x^3+3y}{3x+4y^3}$$

(b) Slope is $y' = -7/7 = -1$, point is $(1,1)$ so the tanget line is $y-1 = -1(x-1)$ or $y = -x+2$.

(c) Differentiate again $12x^2+3y'+3y'3xy''+12x^2y'y'+4y^3y'' = 0$ and plug in $x=y=1$ and $y' = -1$ to get $y'' = -18/7 < 0$. So the curve is concave down at that point.

5. First, $N(12) = 1000 - 240\ln(1) - 240 = 760$ so the point is $(12, 760)$. Then,

$$N'(x) = -\frac{240}{x} - 20$$

so the slope if $N'(12) = -40$. Therefor the tanget line approximation is

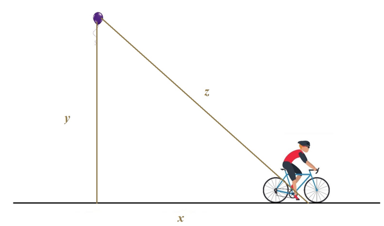
$$N - 760 \approx -40(x - 12)$$

so to increase by 50 pizzas,

$$50 \approx -40(x - 12)$$

so $x \approx 10.75$.

6. Given $dy/dt = 8$ and $dx/dt = 10$, we want dz/dt. From $x^2 + y^2 = z^2$ we get



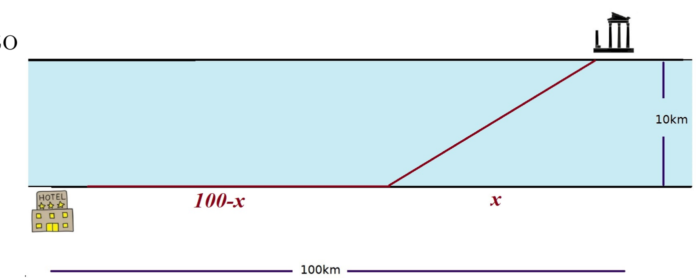
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

After 3 seconds, $x = 30$, $y = 40+24 = 64$ so $z = \sqrt{30^2 + 64^2} = \sqrt{4996}$. So,

$$2 \cdot 30 \cdot 10 + 2 \cdot 64 \cdot 8 = 2\sqrt{4996}\frac{dz}{dt}.$$

gives $\frac{dz}{dt} = \frac{406}{\sqrt{1249}} \approx 11.5$ meters per second.

7. From the picture, the time it takes for them to travel is



$$f(x) = \frac{100 - x}{45} + \frac{\sqrt{10^2 + x^2}}{30}$$

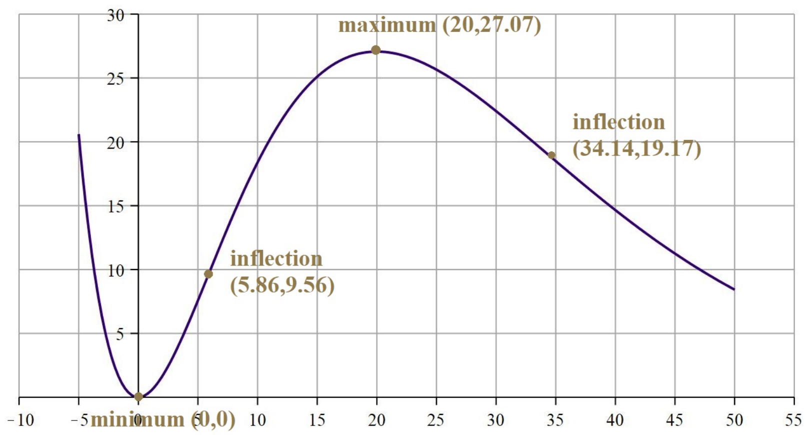
where $0 \le x \le 100$.

The derivative is

$$f'(x) = -\frac{1}{45} + \frac{2x}{30 \cdot 2\sqrt{10^2 + x^2}} = 0$$

which gives $x = 4\sqrt{5}$. We check $f(0) = 46/18 \approx 2.56$, $f(10) = \sqrt{101}/3 \approx 3.35$, and $f(45) \approx 2.47$. So, the shortest time is about 2.47 hours.

8. (a) $f'(x) = e^{-0.1x}x(1 - 0.05x)$ so the function is increasing on $(0, 20)$
   (b) $f''(x) = 0.01e^{-0.1x}(0.5x^2 - 20x + 100)$ so the graph is concave up on $(-5, 20 - 10\sqrt{2})$ and $(20 + 10\sqrt{2}, 50)$.
   (c) Approximately $(34.14, 19.17)$ and $(5.86, 9.56)$.
   (d) Computing: $f(-5) \approx 20.61$, $f(50)8.42$, $f(0) = 0$ (min), $f(20) \approx 27.07$ (max).
   (e) Sketch the graph $y = f(x)$ using the grid below. Clearly label the (x, y) coordinates of endpoints, all critical points, and points of inflection. Make sure your graph matches with the information you provided above.


maximum (20,27.07)
inflection
(34.14,19.17)
inflection
(5.86,9.56)
minimum (0,0)
30
25
20
15
10
5
−10
−5
0
5
10
15
20
25
30
35
40
45
50
55

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