Answers to Math 124 Spring 2022 Final

1. (a) *Factoring*

$$\lim_{\theta\to\pi/2}\frac{\sin^2\theta-5\sin\theta+4}{\sin^2\theta-1}=\lim_{\theta\to\pi/2}\frac{(\sin\theta-1)(\sin\theta-4)}{(\sin\theta-1)(\sin\theta+1)}=\lim_{\theta\to\pi/2}\frac{\sin\theta-4}{\sin\theta+1}=-\frac{3}{2}$$

L'Hospital's Rule

$$\lim_{\theta\to\pi/2}\frac{\sin^2\theta-5\sin\theta+4}{\sin^2\theta-1}=^{\text{LH}}\lim_{\theta\to\pi/2}\frac{2\sin\theta\cos\theta-5\cos\theta}{2\sin\theta\cos\theta}=\lim_{\theta\to\pi/2}\frac{2\sin\theta-5}{2\sin\theta}=-\frac{3}{2}$$

(b) $\lim_{x\to3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{2}=0.$

(c) First,

$$\lim_{x\to0^-}\frac{\sin 3x}{x}=^{\text{LH}}\lim_{x\to0^-}\frac{3\cos 3x}{1}=3$$

and

$$\lim_{x\to0^+}\frac{e^{ax}-1}{x}=^{\text{LH}}\lim_{x\to0^+}\frac{ae^{ax}}{1}=a$$

we have $a=c=3$.

2. (a) $f'(x)=\ln 3\cdot(4\sec^2(4x))3^{\tan(4x)}$

(b) $h'(x)=2x\cos(x^2)\cos^2(3x)-6\sin(x^2)\cos(3x)\sin(3x)$

(c) $y=x^{e^x}$

$$\ln y=e^x\ln x$$
$$\frac{y'}{y}e^x\ln x\frac{e^x}{x}$$
$$y'=e^x\left(\ln x+\frac{1}{x}\right)x^{e^x}$$

(d) $g'(x)=\dfrac{(4e^x+4xe^x)(\sqrt{x^4+5})+4xe^x\left(\frac{4x^3}{2\sqrt{x^4+5}}\right)}{x^4+5}$

3. (a) $\lim_{x\to5}f(x)=\infty$

(b) $\lim_{x\to-3^+}f(x)=-4$

(c) $\lim_{h\to0}\frac{f(h)-f(0)}{h}=f'(0)=2$

(d) $f(1.85)=2(1.85)+2=5.7$.

(e) Let $g'(x)=2f(x)f'(x)=0$ when $f(x)=0$ so $x=-9.5,5,-1$ or when $f'(x)=0$ so when $x=-8,-5,3$.

(f) Let $h'(3)=f'(f(3))f'(3)=f'(4.5)\cdot0=0$.

4. (a) Implicit differentiation:

$$3x^2+3y^2y'=2y+2xy'$$

At $(1,1)$

$$3+3y'=2+2y'$$

so $y'=-1$. The tangent line is

$$y-1=-1(x-1)$$

or $y=-x+2$.

(b) $0.93 \approx -x + 2$ so $x \approx 1.07$.

(c) From $3x^2 + 3y^2y' = 2y + 2xy'$,

$$y' = \frac{2y - 3x^2}{3y^2 - 2x}.$$

The tangent line is horizontal when $y' = 0$ so we solve $2y - 3x^2 = 0$ together with $x^3 + y^3 = 2xy$ to get the point

$$\left(\frac{2^{4/3}}{3}, \frac{2^{5/3}}{3}\right).$$

5. (a) The tangent slope is given by

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{4\pi\cos(\pi t)}{-5\pi\sin(\pi t)} = \frac{4\cos(\pi t)}{-5\sin(\pi t)}$$

so the slope of the normal line is

$$\frac{-1}{dy/dx} = \frac{5\sin(\pi t)}{4\cos(\pi t)}$$

and the equation of the normal line is

$$y - 4\sin(\pi t) = \frac{5\sin(\pi t)}{4\cos(\pi t)}(x - 5\cos(\pi t))$$

(b) Setting $y = 0$ above

$$-4\sin(\pi t) = \frac{5\sin(\pi t)}{4\cos(\pi t)}(x - 5\cos(\pi t))$$

so

$$x_N = \frac{9}{5}\cos(\pi t)$$

(c) $\lim_{t\to 0^+} \frac{9}{5}\cos(\pi t) = \frac{9}{5}$

6. (a) Since $V = b^2h$, differentiating with respect to t:

$$\frac{dV}{dt} = b^2\frac{dh}{dt} + 2bh\frac{db}{dt}.$$

When $b = 4$ and $h = 3$,

$$\frac{dV}{dt} = 4^2(-5) + 2 \cdot 4 \cdot 3(3) = -8.$$

(b) To check whether $\frac{dV}{dt}$ is increasing or decreasing we check the sign of $\frac{d^V}{dt^2}$. So, diffentiate again:

$$\frac{d^2V}{dt^2} = \frac{d}{dt}\frac{dV}{dt} = \frac{d}{dt}\left(b^2\frac{dh}{dt} + 2bh\frac{db}{dt}\right) = b^2\frac{d^2h}{dt^2} + 2b\frac{db}{dt}\frac{dh}{dt} + 2b\frac{dh}{dt}\frac{db}{dt} + 2h\frac{db}{dt}\frac{db}{dt}.$$

Now plug in the values (note that $\frac{d^2h}{dt^2} = 0$ since dh/dt was constant)

$$\frac{d^2V}{dt^2} = 4^2 \cdot 0 + 2b(3)(-5) + 2b(-5)(3) + 2h(3)^2 = -186 < 0$$

so the rate of chage is decreasing.

7. Let (x, y) be the top right corner of the rectangle. Then, from $\frac{x^2}{400} + \frac{y^2}{225} = 1$ we get

$$y = 15\sqrt{1 - \frac{x^2}{20^2}} = \frac{15}{20}\sqrt{400 - x^2} = \frac{3}{4}\sqrt{400 - x^2}$$

so the perimeter is

$$P(x) = 4x + 4y = 4x + 3\sqrt{400 - x^2}$$

with domain $0 \le x \le 20$. To find the critical numbers, we differentiate:

$$P'(x) = 4 + \frac{-3x}{\sqrt{400 - x^2}} = 0$$

to get the only critical number $x = 16$.
To verify it gives a max, either compare $P(0) = 60$, $P(20) = 80$, and $P(16) = 64 + 3\sqrt{144} = 100$ or use the first derivative test on $P'(x)$, for example, $P'(1) = 4 - \frac{3}{\sqrt{399}} < 0$ and $P'(19) = 4 - \frac{57}{\sqrt{39}} > 0$. (Or, if you are up for it, show $P''(16) < 0$).

The dimensions of the rectangle are $2x = 32$ by $2y = 18$.

8. Consider the function $f(x) = \dfrac{8}{x^2 + 4}$.

   (a) There are no x-intercepts since $\frac{8}{x^2+4} = 0$ has no solutions. The y-intercept is $f(0) = 2$.

   (b) Since

   $$\lim_{x\to\infty} \frac{8}{x^2 + 4} = \lim_{x\to-\infty} \frac{8}{x^2 + 4} = 0$$

   $y = 0$ is the horizontal asymptote on both sides.

   (c) The derivative is

   $$f'(x) = -\frac{16x}{(x^2 + 4)^2} = 0$$

   when $x = 0$. Checking its sign with

   $$f'(-1) = -\frac{-16}{(1 + 4)^2} > 0, f'(1) = f'(x) = -\frac{16}{(1 + 4)^2} < 0$$

   we see the function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

   (d) Local max at $x = 0$. No local min.

   (e) The second derivative is

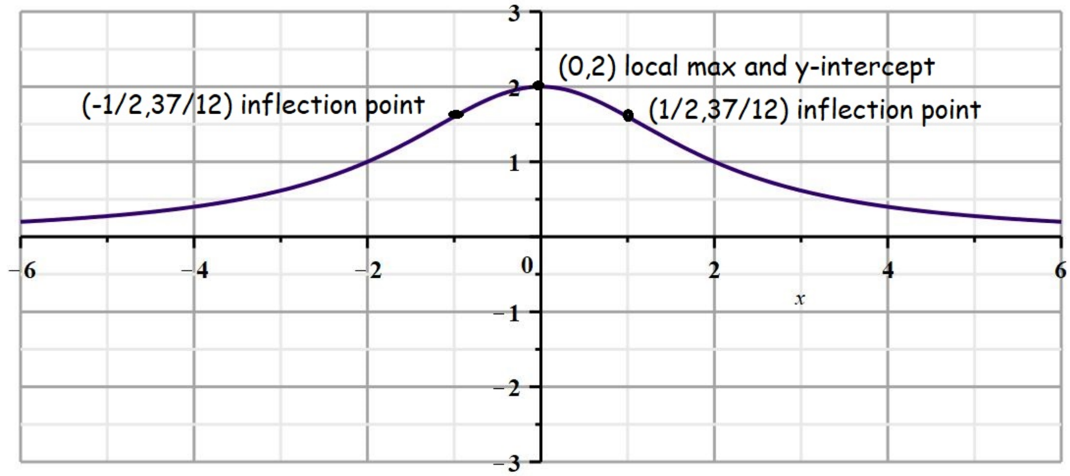
   $$f''(x) = \frac{-16(x^2 + 4) - (-16x) \cdot 2(x^2 + 4)(2x)}{(x^2 + 4)^4} = \frac{16(4x^2 - 1)}{(x^2 + 4)^3} = 0$$

   when $x = \pm 0.5$. Checking signs:

   $$f''(-1) = \frac{16(4 - 1)}{(1 + 4)^3} > 0, f''(0) = \frac{16(-1)}{(4)^3} < 0, f''(1) = \frac{16(4 - 1)}{(1 + 4)^3} > 0$$

   so the graph is concanve up on $(-\infty, -1/2)$ and $(1/2, \infty)$ and concanve down on $(-1/2, 1/2)$. Using $f''(x)$, find the intervals where f is concave up and where it is concave down.

   (f) from above it has inflection at $(1/2, 32/17)$ and $(-1/2, 32/17)$.

(g) 

(0,2) local max and y-intercept
(-1/2,37/12) inflection point
(1/2,37/12) inflection point
x