Solutions to Autumn 2024 Math 124 Final

1. (a) $\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} \cdot \frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3} = \lim_{x\to 2} \frac{4x+1-9}{(x-2)\left(\sqrt{4x+1}+3\right)} = \lim_{x\to 2} \frac{4}{\sqrt{4x+1}+3} = \frac{2}{3}$

   (b) $\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan 2\theta} = \lim_{\theta\to 0} \frac{\sin 3\theta}{\sin 2\theta} \cdot \cos 2\theta = \lim_{\theta\to 0} \frac{\frac{\sin 3\theta}{3\theta}\cdot 3\theta}{\frac{\sin 2\theta}{2\theta}\cdot 2\theta} \cdot \cos 2\theta = \frac{3}{2}$

   (c) $\lim_{x\to\infty} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^{-\frac{3}{2}}} =^{\text{LH}} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(-x^{-2}\right)}{-\frac{3}{2}x^{-\frac{5}{2}}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \cdot \frac{2}{3}x^{1/2} = \infty$

2. (a) $f'(x) = 2\pi \sin(\pi x)\cos(\pi x)\cos(\pi^2 x) - \pi^2 \sin^2(\pi x)\sin(\pi^2 x)$.

   (b) $f'(x) = 2x\sqrt{1+\sqrt{2x}} + \frac{x^2}{2\sqrt{2x}\sqrt{1+\sqrt{2x}}}$

   (c) From $y = (1+\ln x)^{\ln x}$,

$$\ln y = (\ln x)\cdot \ln(1+\ln x)$$

   so

$$\frac{y'}{y} = \frac{\ln x}{x(1+\ln x)} + \frac{\ln(1+\ln x)}{x}$$

   so

$$f'(x) = \left(\frac{\ln x}{x(1+\ln x)} + \frac{\ln(1+\ln x)}{x}\right)\cdot (1+\ln x)^{\ln x}$$

3. (a) $x = -2, -1, 1, 7$

   (b) The derivative $k'(x) = f'(x)\cdot g'(f(x))$ does not exist if $f'(x)$ does not exist or if $g'(f(x))$. So at $x = 7$ or $x = 11$.

   (c) $g'(12) = 1/3$ (d) $\lim_{x\to 3^+} (f(g(x))) = 3$

   (e) $h'(3) = -13/96$ and $\lim_{x\to 11^+} h(x) = -\infty$

   (f) $-2/3$ (g)=2/3.

4. A curve is given implicitly by the equation

$$2(x^2+y^2)^2 = 25(x^2-y^2).$$

   (a) Differentiate:

$$4(x^2+y^2)\cdot(2x+2yy') = 50x - 50yy'$$

   When $x = 3$ and $y = 1$,

$$4(9+1)\cdot(6+2y') = 150 - 50y'$$

   so $y' = -\frac{9}{13}$ and the tangent line is

$$y - 1 = -\frac{9}{13}(x-3).$$

(b) Differentiating again

$$4(2x+2yy')\cdot(2x+2yy')+4(x^2+y^2)\cdot(2+2y'y'+2yy'')=50-50y'y'-50yy''$$

when $x=3$, $y=1$, $y'=-\frac{9}{13}$ we get

$$4\left(6+2\left(-\frac{9}{13}\right)\right)\cdot\left(6+2\left(-\frac{9}{13}\right)\right)+4(9+1)\cdot\left(2+2\left(-\frac{9}{13}\right)^2+2y''\right)=50-50\left(-\frac{9}{13}\right)^2-50y''$$

or

$$4\left(\frac{60}{13}\right)^2+80\left(\frac{250}{169}+y''\right)=50-50\frac{81}{169}-50y''$$

so

$$130y''=\frac{8450}{169}-\frac{4050}{169}-\frac{20000}{169}-\frac{14400}{169}=-\frac{30000}{169}$$

so $y''=-\frac{3000}{2197}$.

5. (a)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-2}{2+\frac{1}{t}}$$

with value of 1/3 when $t=1$. So, the tangent line is $y+1=\frac{1}{3}(x-2)$ and linearization is $f(x)\approx -1+\frac{1}{3}(x-2)$, Then,

$$y=f(1.9)\approx -1+\frac{1}{3}(1.9-2)=-\frac{31}{30}\approx -1.033$$

(b)

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{\frac{d}{dt}\frac{dy}{dx}}{dx/dt}=\frac{\frac{6t\cdot\left(2+\frac{1}{t}\right)-(3t^2-2)\cdot\left(-\frac{1}{t^2}\right)}{\left(2+\frac{1}{t}\right)^2}}{2+\frac{1}{t}}$$

which has a value of 19/27 at $t=1$. The 2nd derivative is positive at $t=1$. This means the tangent line is below the curve. We conclude that our approximation is an under-estimate.

6. Given dx/dt, we want dA/dt. Let h be the height. Then

$$5^2+h^2=x^2$$

so

$$A=\frac{1}{2}\cdot 10\cdot h=5\sqrt{x^2-25}$$

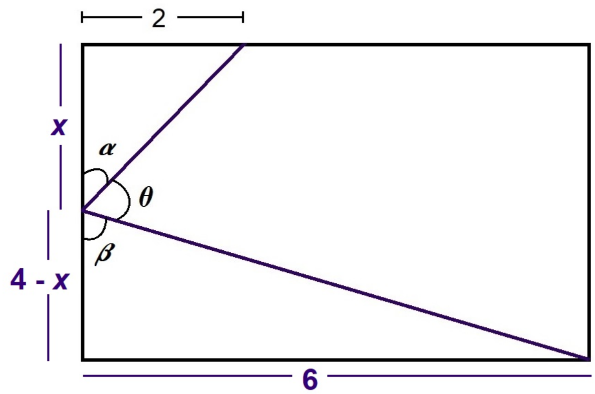
which gives

$$\frac{dA}{dt}=5\frac{5}{2\sqrt{x^2-25}}\cdot 2x\cdot\frac{dx}{dt}$$

using the values

$$\frac{dA}{dt}=5\frac{5}{2\sqrt{169-25}}\cdot 2\cdot 13\cdot 3=\frac{65}{4}.$$

7. To determine the location of P, name the variable x. Then,



The angles are given by

$$\alpha = \tan^{-1}\frac{2}{x}, \quad \beta = \tan^{-1}\frac{6}{4-x}$$

so

$$\theta = \pi - \alpha - \beta = \pi - \tan^{-1}\frac{2}{x} - \tan^{-1}\frac{6}{4-x}.$$

Differentiate and set equal to 0:

$$\theta' = -\frac{1}{1+\left(\frac{2}{x}\right)^2}\cdot\left(\frac{-2}{x^2}\right) - \frac{1}{1+\left(\frac{6}{4-x}\right)^2}\cdot\left(\frac{6}{(4-x)^2}\right) = \frac{2}{x^2+4} - \frac{6}{(4-x^2)+36} = 0$$

simplifying

$$x^2 + 4x - 20 = 0$$

which gives $x = -2 + 2\sqrt{6} \approx 2.899$ (the other value is negative). Check, for example,

$$\theta'(1) = \frac{2}{5} - \frac{6}{45} = \frac{12}{45} > 0, \quad \theta'(3) = \frac{2}{13} - \frac{6}{37} = \frac{54-78}{13\cdot 37} < 0$$

to conclude we have a max. (Computing $\theta''(-2+2\sqrt{6})$ would have been messy.)

8. $f(x) = 3xe^{-x^2/6}$.

(a) Solve $f(x) = 0$ to get $x = 0$. The y-intercept (and the x-intercept) is $(0, 0)$.

(b) There are no vertical intercepts: $f(x)$ is defined for all x.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3x}{e^{x^2/6}} =^{\text{LH}} \frac{3}{\frac{1}{3}xe^{x^2/6}} = 0$$

so $y = 0$ is a horizontal asymptote, in both directions.

(c) Setting $f'(x) = 3e^{-x^2/6} - 3x \cdot \frac{1}{3}xe^{-x^2/6} = \left(3 - x^2\right) e^{-x^2/6} = 0$ gives $x = -\sqrt{3}, \sqrt{3}$.

(d) Use test values $x = -2, 0, 2$ to see that $f'(0) > 0$ and $f'(\pm 2) < 0$. (Other test values are OK.) So, $f(x)$ is increasing if $-\sqrt{3} < x < \sqrt{3}$. It is decreasing if $x < -\sqrt{3}$ or $x > \sqrt{3}$.

(e) The work above shows that $f(x)$ has a relative minumum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$. The points are $(-\sqrt{3}, -3\sqrt{3}e^{-1/2})$ and $(\sqrt{3}, 3\sqrt{3}e^{-1/2})$.
Note that $\sqrt{3} \approx 1.73$ and $3\sqrt{3}e^{-1/2} \approx 3.15$

(f) $f''(x) = e^{-x^2/6} - (3 - x^3) \cdot \frac{1}{3}xe^{-x^2/6} = \frac{1}{3}x\left(x^2 - 9\right) e^{-x^2/6} = 0$ gives $x = -3, 0, 3$. Use test values $x = -4, -1, 1, 4$ to see that $f''(x) < 0$ if $x \in (-\infty, -3) \cup (0, 3)$(this is where f is concave down) and $f''(x) > 0$ if $x \in (-3, 0) \cup (3, \infty)$ (this is where f is concave up)

(g) The sign of $f''(x)$ changes at $x = -3, 0, 3$ from above. The inflection points are $(-3, -9e^{-3/2})$, $(0, 0)$ and $(3, 9e^{-3/2})$. Note that $9e^{-3/2} \approx 2.008$.

(h) The graph with extrema and inflection points marked: