Answers to Math 124 Autumn 2023 Final Exam

1. (a) $\lim_{x\to 2}\dfrac{\sqrt{x^2+5}-3}{x^2-2x}\cdot\dfrac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}=\lim_{x\to 2}\dfrac{x^2-4}{(x^2-2x)\left(\sqrt{x^2+5}+3\right)}=\lim_{x\to 2}\dfrac{x+2}{x\left(\sqrt{x^2+5}+3\right)}=\dfrac{1}{3}$

   (b) $\lim_{x\to\pi/2}\dfrac{1-\sin x}{x-\pi/2}=^{\mathrm{LH}}\lim_{x\to\pi/2}\dfrac{-\cos x}{1}=0$

   (c) From $y=(\sin x)^{\tan x}$,

   $$\ln y=(\tan x)\left(\ln(\sin x)\right)=\cos x\cdot\frac{\ln(\sin x)}{\sin(x)}.$$

   Since

   $$\lim_{x\to 0+}\ln y=\lim_{x\to 0+}(\cos x)\cdot\frac{\ln(\sin x)}{\sin(x)}=-\infty$$

   $\lim_{x\to 0^+} y=0$

2. (a) $f(x)=\dfrac{x\sin(x^3)}{1+e^{4x}}$

   $f'(x)=\dfrac{\left(\sin(x^3)+x\cdot 3x^2\cdot\cos(x^3)\right)(1+e^{4x})-\left(x\sin(x^3)\right)(4e^{4x})}{(1+e^{4x})^2}$

   (b) $g(x)=\tan\left(a+\ln\left(b+e^{cx}\right)\right)$ where a,b,c are constants.

   $g'(x)=\sec^2\left(a+\ln\left(b+e^{cx}\right)\right)\cdot\dfrac{ce^{cx}}{b+e^{cx}}$

   (c) $k(x)=\arctan\left(\dfrac{2x-1}{4+7x}\right)$

   $k'(x)=\dfrac{1}{1+\left(\frac{2x-1}{4+7x}\right)^2}\cdot\left(\dfrac{2(4+7x)-4(2x-1)}{(4+7x)^2}\right)$

3. (a) $f'(0)=-1$

   (b) $w'(7)=-8$

   (c) $u'(0)=-\dfrac{4}{5}$

   (d) $x=3,-4.6.5$

   (e) $(1,3)$ and $(4,\infty)$

4. Implicit differentiation:

   $$4(x^2+y^2)(2x+2yy')=25(2x-2yy').$$

   so

   $$y'=\frac{50x-8x(x^2+y^2)}{50y+8y(x^2+y^2)}$$

   tangent is vertical when the denominator equals zero:

   $$0=50y+8y(x^2+y^2)=y\left(50+8(x^2+y^2)\right)$$

   since $50+8(x^2+y^2)>0$ we must have $y=0$ so using the curve equation

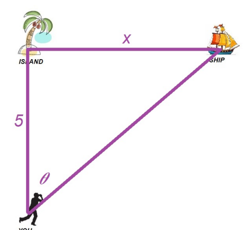
   $$2(x^2)^2=25(x^2).$$

   so $x^2(2x^2-25)=0$. Since $x\neq 0$, we get $x=\pm\sqrt{5/2}$ so two points $(\pm\sqrt{5/2},0)$.

5. (a) From
$$\frac{dy}{dx} = \frac{3t^2+8t}{e^t+1}$$
the tangent is horizontal when $3t^2+8t=0$ so $t=0$ or $t=-8/3$. The tangent is never vertical since $e^t+1 \neq 0$.

(b) From
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{3t^2+8t}{e^t+1}}{\frac{dx}{dt}} = \frac{e^t(-3t^2-2t+8)+6t+8}{(e^t+1)^3}$$
at $t=0$
$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{1(8)+8}{(2)^3} > 0$$
so concave up and at $t=-8/3$
$$\left.\frac{d^2y}{dx^2}\right|_{t=-3/8} = \frac{e^{-3/8}(-56/8)-8}{(e^{-3/8}+1)^3} < 0$$
so concave down.

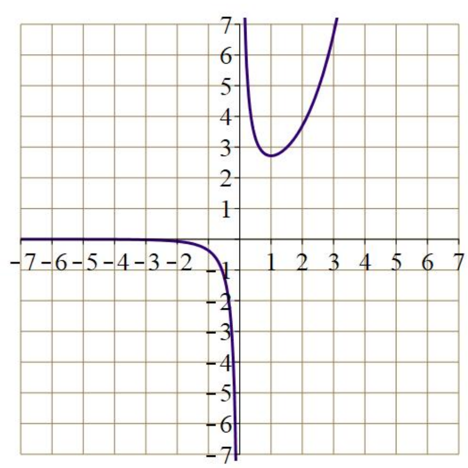
6. Differentiate $\tan\theta = x/5$ and plug in values:



$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt} \quad \text{so} \quad \sec^2(\pi/6)\frac{5}{4} = \frac{1}{5}\frac{dx}{dt}$$
to get $dx/dt = 25/3$ miles per hour.

7. Using similar triangles $b = \dfrac{2a}{a-1}$ so the area of the triangle is $A = \dfrac{a^2}{a-1}$ with domain $a > 1$. From
$$A' = \frac{a^2-2a}{(a-1)^2}$$
we get the critical number $a=2$ (since $a=0$ is not possible). Then $b=4$. To check it is minimum:
$$A'' = \frac{(2a-2)(a-1)^2-(a^2-2a)\cdot 2(a-1)}{(a-1)^4} = \frac{-2}{(a-1)^2} < 0$$

8. (a) $\displaystyle\lim_{x\to0^-}\frac{e^x}{x} = -\infty$ and $\displaystyle\lim_{x\to0^+}\frac{e^x}{x} = \infty$



(b) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x} =^{\text{LH}} \lim_{x\to\infty}\frac{e^x}{1} = \infty$ and $\displaystyle\lim_{x\to-\infty}\frac{e^x}{x} = 0$ so $y=0$ is a horizontal asymptote on the left as $x\to-\infty$.

(c) From $f'(x) = \dfrac{e^x(x-1)}{x^2}$ the function is decreasing on $(-\infty,0)$ and $(0,1)$.

(d) From
$$f'(x) = \frac{e^x(x^2-2x+2)}{x^3}$$
the graph is concave up for $x>0$ and concave down for $x<0$.