Answers to Math 124 Autumn 2022 Final Exam

1. (a) $\lim_{x\to 0}\frac{\cos(3x)-1}{x^2} =^{\text{LH}} \lim_{x\to 0}\frac{-3\sin(3x)}{2x} =^{\text{LH}} \lim_{x\to 0}\frac{-9\cos(3x)}{2} = -\frac{9}{2}$

   (b) $\lim_{x\to 0^+}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \infty$ because $\lim_{x\to 0^+}(e^x+e^{-x}) = 2$, $\lim_{x\to 0^+}(e^x-e^{-x}) = 0$, and $e^x-e^{-x} > 0$ when $x > 0$.

   (c) Since
   $$\lim_{x\to\infty}\frac{2x+1}{x+1} =^{\text{LH}} \lim_{x\to\infty}\frac{2}{1} = 2,$$
   $$\lim_{x\to\infty}(\ln(2x+1)-\ln(x+1)) = \lim_{x\to\infty}\ln\left(\frac{2x+1}{x+1}\right) = \ln 2$$

2. (a) $f'(x) = \dfrac{e^{3x}\left(\frac{2}{1+4x^2}+2x\right) - 3e^{3x}\left(\arctan(2x)+x^2\right)}{e^{6x}}.$

   (b) $g'(x) = \cos(x^4+e^{x^2+x})\cdot\Big(4x^3+e^{x^2+x}(2x+1)\Big).$

   (c) $h' = (x) = \frac{1}{x}\cdot\arctan(4x^2)\cdot\sqrt{x^2+x} + \ln(x)\cdot\frac{8x}{1+16x^4}\cdot\frac{2x+1}{2\sqrt{x^2+x}} + \ln(x)\cdot\arctan(4x^2)\cdot\sqrt{x^2+x}.$

3. By the Chain Rule, $g'(x) = f'(f(x))\cdot f'(x)$ and $h'(x) = 2f(x)f'(x)$.

   (a) $f'(2.5) = -1$.

   (b) $g'(5) = f'(f(5))\cdot f'(5) = f'(0)\cdot\frac{1}{2} = -\frac{1}{2}$.

   (c) $g'(x) \neq 0$ because $f'(x) \neq 0$ for any x. Since, $f'(3)$ is not defined, g' is not defined when $x = 3$ or $f(x) = 3$. So the three critical points of g where g' is not defined are: $x = 3, -1, 11$. g is differentiable everywhere else.

   (d) $\lim_{x\to 0}\frac{g(x)}{x} =^{\text{LH}} \lim_{x\to 0}\frac{g'(x)}{1} = g'(0) = f'(f(0))\cdot f'(0) = -1\cdot -1 = 1.$

   (e) $h'(x) > 0$ when both f, f' are positive or both are negative: $(2,3)$ and $(5,\infty)$.

4. (a) $1^{2\cdot 1} = 1^1 = 1$

   (b) $\ln y = \ln\left(x^{2y}\right) = 2y\ln x$
   $$\frac{y'}{y} = 2y'\ln x + \frac{2y}{x}$$
   WHen $x = y = 1$,
   $$\frac{y'}{1} = 2y'\ln 1 + \frac{2}{1} = 2$$
   Tangent line is $y-1 = 2(x-1)$ so the approximations is $y - 1 \approx 2(1.05-1)$, so $y \approx 1.1$.

   (c) Second derivative:
   $$\frac{y''y - y'y'}{y^2} = 2y''\ln x + 2y'\cdot\frac{1}{x} + \frac{2y'x-2y}{x^2}$$
   When $x = y = 1$, $y' = 2$,
   $$\frac{y''-4}{1} = 2y''\ln 1 + 4 + 2 = 6$$
   so $y' = 10 > 0$, the curve is concave up and we have an underestimate.

5.
$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2 + 4t - 1, \frac{dy}{dx} = \frac{3t^2 + 4t - 1}{2t}$$

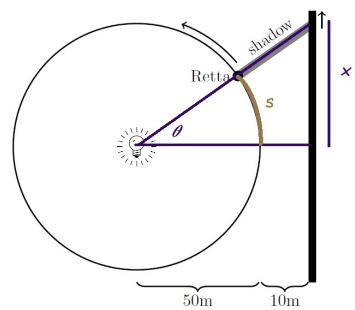
(a)
$$m_1 = \frac{3+4-1}{2} = 3, m_{-1} = \frac{3-4-1}{-2} = 1$$

(b) $(x(1), y(1)) = (1, 2) = (x(-1), y(-1))$

(c) It is the first picture with both slopes positive.

(d) On the steeper piece (near $t = 1$ with slope 3), you are going up $\frac{dy}{dt} > 0$ and right $\frac{dy}{dt} > 0$. On the other piece (near $t = -1$ with slope 1), you are going down $\frac{dy}{dt} < 0$ and left $\frac{dy}{dt} < 0$.

6. We know $\frac{ds}{dt} = 4$ and we want $\frac{dx}{dt}$.Since $\tan\theta = x/60$ and $s = 50\theta$ the relation is
$$\tan\left(\frac{s}{50}\right) = \frac{x}{60}$$
so
$$\sec^2\left(\frac{s}{50}\right) \cdot \frac{ds}{dt} \cdot \frac{1}{50} = \frac{1}{60} \cdot \frac{dx}{dt}$$
When Retta is 35m from wall, $\theta = \pi/3$ and $s = 50\pi/3$ so
$$\frac{dx}{dt} = \frac{96}{5} = 19.2.$$



7. There are several ways to find the distance s he swims from A to B. Using Law of Cosines with the triangle AOB:

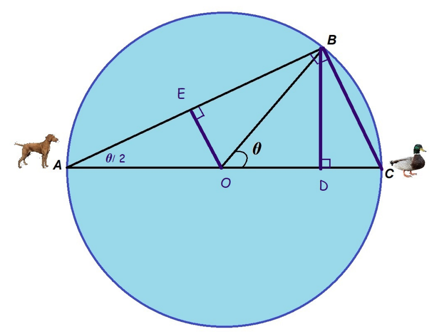
$$s^2 = 120^2 + 120^2 - 2 \cdot 120 \cdot 120 \cdot \cos(\pi - \theta) = 120^2(2 + 2\cos\theta)$$

Using the right triangle ABD:

$$s^2 = (120\sin\theta)^2 + (120 + 120\cos\theta)^2 = 120^2(2 + 2\cos\theta)$$

Using the right triangle ABC :
$$s = 240\cos(\theta/2)$$

Using the right triangle AED:
$$s = 2 \cdot 120\cos(\theta/2).$$



So you use one of

$$f(\theta) = \frac{120\sqrt{2 + 2\cos\theta}}{6} + \frac{120\theta}{2} \quad \text{or} \quad g(\theta) = \frac{240\cos(\theta/2)}{6} + \frac{120\theta}{2}$$

with $0 \le \theta \le \pi$. The critical number is $\theta = \pi/3$, but the minimum happens at the endpoint $\theta = \pi$, so he should run and not get wet at all (which he prefers).

8. Consider the function $f(x) = -\frac{1}{6}x^2 + x - \frac{2}{3}\ln x$.

(a) $x > 0$ (because of $\ln x$)

(b) Vertical $x = 0$ (because of $\ln x$). No horizontal since $\lim_{x\to\infty} f(x) = \infty$.

(c) $f'(x) = -\dfrac{1}{3}x + 1 - \dfrac{2}{3x} = -\dfrac{(x-1)(x-2)}{3x}$. CN: $x = 1, 2.$($x = 0$ in not a CN because it is not in the domain.)

(d) $f(x)$ is increasing on $(1, 2)$ and decreasing on $(0, 1)$ and $(2, \infty)$.

(e) Min at $x = 1$ with point approximately $(1, 0.83)$. Max at $x = 2$ with point approximately $(2, 0.87)$.

(f) $f''(x) = -\dfrac{1}{3} + \dfrac{2}{3x^2} = -\dfrac{(x-\sqrt{2})(x+\sqrt{2})}{3x^2}$. Concave up on $(0, \sqrt{2})$ and concave down on $(\sqrt{2}, \infty)$. (Note that the domain is $x > 0$). Inflection point $(\sqrt{2}, f(\sqrt{2})) \approx (1, 41, 0.85)$.

(g)