Winter 2022

MATH 120 Final Exam

Final Exam: 5:00-7:50pm, Saturday March 12, @ARC 147

Name: Quiz section:

7-digit UW ID:

Exam Instruction:

- You have 170 minutes to complete 7 questions. Distribute your time wisely.
- Show your work to earn full credit. If you can not completely solve a problem, providing reasonable work and steps may earn you some partial credit.
- Box $\boxed{\text{your final answer}}$ for each problem.
- The last two pages are scratch papers, tear them off and do NOT turn in unless you have written down additional work to be graded.
- **Do NOT write within 1 cm of the edge.** Your exam will be scanned for grading. If you run out of space, specify "see scratch paper", then write you additional work on the scratch paper and turn it in together with your exam.
- You can prepare one hand-written double-sided 8.5" × 11" page of notes and bring it to the exam.
- You may use a basic calculator that can NOT graph or solve equation (e.g., TI-30X IIS). All other electronic devices (e.g., cell phone, earbuds) should be turned off and put away during the exam.
- You must finish the exam independently. **Giving or receiving any assistance on the exam is considered cheating, which will result in a grade of zero for the exam.** Do not discuss the exam questions with other students before the exam grade is released.

1. (14 pts) When a cherry farm has 150 cherry trees planted, each tree yields 40 pounds of cherries per year. For every 10 additional trees planted in the farm, the annual yield per tree will decrease by 2 pounds due to overcrowding.

(a) Let x be the number of trees planted in the farm. Write the annual yield of cherries per tree (in pound) as a linear function of x.

x = # of trees. y = annual yield per tree

$$\left.\begin{array}{l} \text{slope} = \dfrac{\Delta y}{\Delta x} = \dfrac{-2}{10} = -\dfrac{1}{5} \\ \text{pt}: (x=150,\ y=40) \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} y = -\frac{1}{5}(x-150)+40 \\ \text{or } y = -\frac{1}{5}x + 70 \end{array}}$$

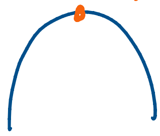
(b) Find the function $f(x)$ that represents the total annual yield from all the cherry trees in the farm. Then find the number of trees $x = c$ that maximizes the total annual yield $f(x)$.

total yield = yield per tree $\times$ # of trees

$$f(x) = y \cdot x = \left(-\frac{1}{5}x + 70\right)\cdot x = -\frac{1}{5}x^2 + 70x$$

$$\text{max @ } x = h = \frac{-b}{2a} = \frac{-70}{2(-\frac{1}{5})} = 175 \quad \boxed{c = 175}$$

(h, k)



$$f(h) = k = c - \frac{b^2}{4a} = 0 - \frac{70^2}{-\frac{4}{5}} = 6125$$

(c) Find the inverse function of $f(x)$ restricted on the domain $c \le x \le 250$ where c is the number you found in part (b). Specify the domain and range of $f^{-1}(x)$.

$f(250) = 5000$

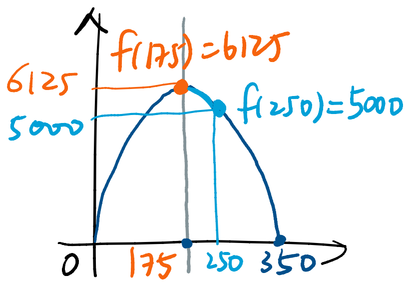
$f(175) = 6125$

$$y = -\frac{1}{5}x^2 + 70x \Rightarrow -\frac{1}{5}x^2 + 70x - y = 0$$

$$\Rightarrow x = \frac{-70 \pm \sqrt{70^2 - \frac{4}{5}y}}{2(-\frac{1}{5})} \quad \text{or} \quad x = 175 \pm \sqrt{175^2 - 5y}$$

Since f has domain $[175, 250]$

$\Rightarrow$ choose the + branch ($x \ge 175$)



$$\boxed{\begin{array}{l} f^{-1}(x) = 175 + \sqrt{175^2 - 5x} \\ \text{range of } f^{-1} = \text{domain of } f = [175, 250] \\ \text{domain of } f^{-1} = \text{range of } f : [5000, 6125] \end{array}}$$

2. (16 pts) In 2010, there were 30 raccoons (trash pandas) in a neighborhood. The raccoon population grew exponentially to reach 70 in 2016. Then between 2016 and 2020, the raccoon population increased by 12% every year.

In 2010, there were 20 trash cans in the neighborhood. Between 2010 and 2016, the number doubled every three years. Then between 2016 and 2020, the number of trash cans stayed unchanged.

Let t be the number of years since 2010.

(a) Express the number of raccoons between 2010 and 2020 as a piecewise (multi-part) function of t.

2010–2016: $P(t) = 30(b^t)$ plug in $(t=6, P(t)=70)$

$70 = 30b^6 \Rightarrow b = \left(\frac{7}{3}\right)^{\frac{1}{6}}$

$$P_A(t) = \begin{cases} 30\left(\frac{7}{3}\right)^{\frac{t}{6}} & 0 \le t \le 6 \\ 70(1.12)^{t-6} & 6 < t \le 10 \end{cases}$$

(b) Express the number of trash cans between 2010 and 2020 as a piecewise (multi-part) function of t.

2010–2016: $P(t) = 20(2^{\frac{t}{3}})$, @ 2016, $P(6) = 20(2^{\frac{6}{3}}) = 80$

$$P_B(t) = \begin{cases} 20(2)^{\frac{t}{3}} & 0 \le t \le 6 \\ 80 & 6 < t \le 10 \end{cases}$$

(c) Between the year 2010 and 2020, find all the time t when there were the same number of raccoons and trash cans. **Leave your answers in exact form, do *not* use calculator to round your answers to decimal numbers.**

Solve t from $P_A(t) = P_B(t)$

when $0 \le t \le 6$, $30\left(\frac{7}{3}\right)^{\frac{t}{6}} = 20(2)^{\frac{t}{3}}$

$\Rightarrow \frac{30}{20} = \left(\frac{2^{\frac{1}{3}}}{\left(\frac{7}{3}\right)^{\frac{1}{6}}}\right)^t \Rightarrow \boxed{t_1 = \frac{\ln\left(\frac{3}{2}\right)}{\ln\left(\frac{2^{\frac{1}{3}}}{\left(\frac{7}{3}\right)^{\frac{1}{6}}}\right)}}$ $\left(\begin{matrix}\approx 4.51 \\ 0 < t_1 < 6 \ \checkmark\end{matrix}\right)$

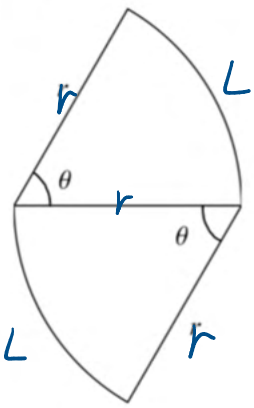
or $\frac{6\ln(3/2)}{\ln(12/7)}$

when $6 < t < 10$, $70(1.12)^{t-6} = 80$

$\Rightarrow (1.12)^{t-6} = \frac{80}{70} \Rightarrow \boxed{t_2 = \frac{\ln\left(\frac{8}{7}\right)}{\ln(1.12)} + 6}$ $\left(\begin{matrix}\approx 7.18 \\ 6 < t_2 < 10 \ \checkmark\end{matrix}\right)$

3. (15 pts) You have 100 meters of fencing materials to enclose two identical regions of circular wedge as illustrated in the picture below. Find the radius r (in meter) and the angle θ (in radiance) of the circular wedge so that the total fenced area is maximized?

(Note that the line of radius shared by the two circular wedges *is* part of the fence. The total fenced area is referring to the total area of two identical circular wedges.)



Constraint: total length $=100$

$$3r + 2L = 3r + 2r\theta = 100$$

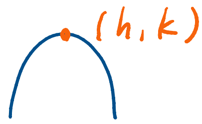
$$\Rightarrow \theta = \frac{100-3r}{2r} = \frac{50}{r} - \frac{3}{2}$$

maximize: area $= 2 \times \frac{1}{2} r^2 \theta$

$$= r^2 \theta$$

$$= r^2 \left(\frac{100-3r}{2r} \right)$$

$$A(r) = 50r - \frac{3}{2} r^2$$



max @ $r = h = \frac{-b}{2a} = \frac{-50}{2(-\frac{3}{2})} = \frac{50}{3}$ meter

$$\theta = \frac{50}{r} - \frac{3}{2} = \frac{50}{50/3} - \frac{3}{2} = 3 - \frac{3}{2} = \frac{3}{2} \text{ rad}$$

$$\boxed{r = \frac{50}{3},\ \theta = \frac{3}{2}}$$

4. (15 pts) Consider the linear-to-linear rational function $f(x) = \dfrac{4x-6}{x-2}$: $= \dfrac{ax+b}{cx+d}$

(a) Identify the x–intercept, y–intercept, vertical asymptote and horizontal asymptote of $f(x)$.

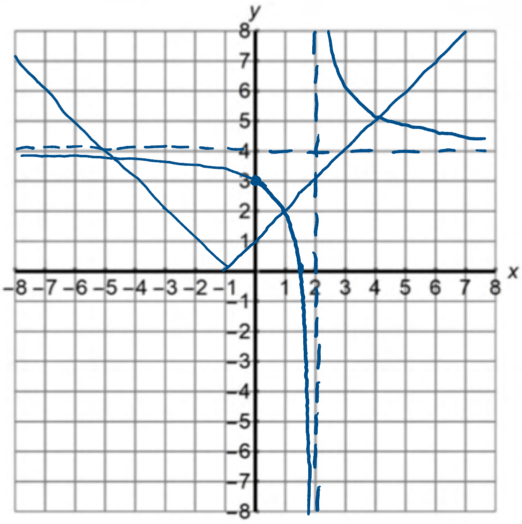
x-int: solve $f(x) = \dfrac{4x-6}{x-2} = 0 \Rightarrow x = \dfrac{6}{4} = \dfrac{3}{2}$ $\left(\dfrac{3}{2}, 0\right)$

y-int: $f(0) = \dfrac{-6}{-2} = 3$ $(0, 3)$

V-A: $x = 2$

H-A: $y = \dfrac{a}{c} = \dfrac{4}{1} = 4$

(b) Briefly sketch the graph of $f(x)$ on the xy–plane below, your graph should address what you found in (a).



$|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \ (x \geq -1) \\ -(x+1) & \text{if } x+1 < 0 \ (x < -1) \end{cases}$

(c) Sketch the graph of $g(x) = |x+1|$ on the xy–plane above. Then solve the equation $f(x) = g(x)$. Leave your answer in exact form.

When $x \geq -1$, solve $\dfrac{4x-6}{x-2} = x+1$

$\Rightarrow 4x-6 = (x+1)(x-2) \Rightarrow 4x-6 = x^2-x-2$

$x^2-5x+4=0 \Rightarrow (x-1)(x-4)=0 \Rightarrow \boxed{x=1,\ x=4}$ (both ≥ -1)

When $x < -1$, solve $\dfrac{4x-6}{x-2} = -(x+1)$

$\Rightarrow 4x-6 = -(x+1)(x-2) \Rightarrow 4x-6 = -(x^2-x-2)$

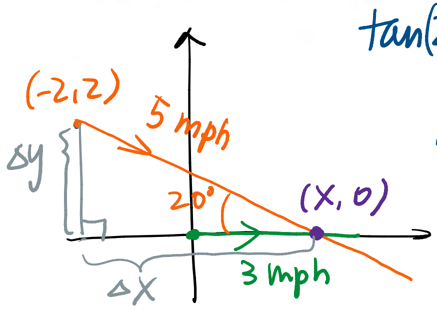
$x^2+3x-8=0 \Rightarrow x = \dfrac{-3 \pm \sqrt{9+32}}{2}$, $x = \dfrac{-3+\sqrt{41}}{2}$ (> -1 ✗)

$\boxed{x = \dfrac{-3-\sqrt{41}}{2}}$ (< -1 ✓)

5. (15 pts) A sea lion is chilling on a buoy 2 miles due north and 2 miles due west of a person in a kayak. At noon, the person starts paddling due east at a speed of 3 mph, and the sea lion starts swimming due southeast at a speed of 5 mph. The path of the sea lion makes an angle of 20° with the path of the person. Impose a coordinate system with the origin being the person's starting position.

For each of the following questions, round the numbers in your final answer to have 1 decimal place.

(a) Find the coordinate of the point of intersection between the path of the kayaking person and the path of the swimming sea lion.



$$\tan(20°) = \frac{\Delta y}{\Delta x} \Rightarrow \Delta x = \frac{\Delta y}{\tan(20°)} = \frac{2}{\tan(20°)}$$

$$x = -2 + \Delta x = -2 + \frac{2}{\tan(20°)} \approx -2 + 5.5 = 3.5$$

$$\boxed{(3.5,\ 0)}$$

(b) Find the parametric equations for the x and y coordinate of the sea lion t hours after noon.

$$x(t) = x_0 + v_x t = -2 + v\cos(20°)t \qquad (v = 5\text{ mph})$$
$$= -2 + 5\cos(20°)t \Rightarrow \boxed{x(t) = -2 + 4.7t}$$

$$y(t) = y_0 + v_y t = 2 - v\sin(20°)t$$
$$= 2 - 5\sin(20°)t \Rightarrow \boxed{y(t) = 2 - 1.7t}$$

(c) Use your answer for part (b) to find the distance between the person and the sea lion at 4 pm.

person: $x(t) = 3t$
$y(t) = 0$

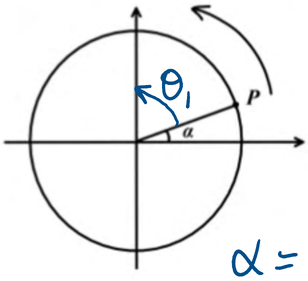
$$\text{dist} = \sqrt{(-2+4.7t-3t)^2 + (2-1.7t)^2}$$
$$= \sqrt{(-2+1.7t)^2 + (2-1.7t)^2}$$

@ $t = 4$. $\text{dist} = \sqrt{2(2-1.7\times 4)^2} \approx \boxed{6.8}$

6. (14 pts) A Ferris wheel rotates at a constant speed of 6 feet per second. It takes 0.8 minute to complete one revolution. A coordinate system is imposed so that the center of the wheel is the origin and the wheel is rotating counterclockwise. At $t = 0$, a butterfly lands at point P referenced by the angle α in the picture, it reaches the top of the ride in 5 seconds.

For each of the following questions, round the numbers in your final answer to have 2 decimal places.

(a) Find the radius r (in ft). Find the angle α (in radiance).



$V = 6$ ft/sec

$T = 0.8\text{min} = 48\text{ sec} \Rightarrow \omega = \frac{2\pi}{48} = \frac{\pi}{24}$ rad/sec ≈ 0.13

$r = \frac{V}{\omega} = \frac{6}{\pi/24} = \frac{144}{\pi}$

$\alpha = \frac{\pi}{2} - \theta_1 = \frac{\pi}{2} - \omega t$

$= \frac{\pi}{2} - \frac{\pi}{24} \times 5 = \frac{7\pi}{24}$

$\boxed{r \approx 45.84 \text{ ft} \quad \alpha \approx 0.92 \text{ rad}}$

(b) Find the parametric equations $x(t)$ and $y(t)$ for the coordinate of the butterfly t seconds later.

initial angle $\phi = \alpha = 0.92$

$x(t) = r\cos(\phi + \omega t)$

$y(t) = r\sin(\phi + \omega t)$

or $\frac{\pi}{24}$

$\boxed{\begin{aligned} x(t) &= 45.84\cos(0.92 + 0.13t) \\ y(t) &= 45.84\sin(0.92 + 0.13t) \end{aligned}}$

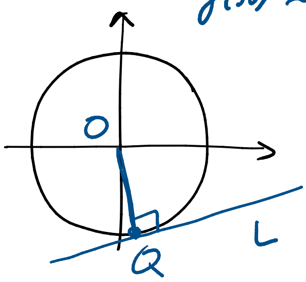
(c) After 30 seconds, the butterfly flies away along a path tangent to the wheel. Find the equation of the path.

$x(30) \approx 4.9234$

$y(30) \approx -45.5748$

(if used $\omega = \frac{\pi}{24}$, $x(30) \approx 6.1515$

$y(30) \approx -45.4254$)



tangent line L is $\perp OQ$

slope of OQ: $m_1 = \frac{\Delta y}{\Delta x} = \frac{y(30)}{x(30)}$

slope of L: $m_2 = -\frac{1}{m_1} = -\frac{x(30)}{y(30)} \approx 0.11$ (≈ 0.14 if $\omega = \frac{\pi}{24}$)

line: $\boxed{y = 0.11(x - 4.92) - 45.57}$

($y = 0.14(x - 6.15) - 45.43$ if used $\omega = \frac{\pi}{24}$)

7. (11 pts) A spacecraft landed on a planet whose temperature fluctuates periodically. 30 hours after the landing, the minimum temperature of 4°F was recorded the first time. 72 hours after the landing, the maximum temperature of 120°F was recorded the first time.

(a) Find a sinusoidal function $f(t) = A\sin\left[\frac{2\pi}{B}(t-C)\right] + D$ which models the temperature t hours after the landing.

$y_{min} = 4°F$ @ $t_{min} = 30$

$y_{max} = 120°F$ @ $t_{max} = 72$

$A = \frac{y_{max} - y_{min}}{2} = \frac{120-4}{2} = 58$

$D = \frac{y_{max} + y_{min}}{2} = \frac{120+4}{2} = 62$

$\frac{B}{2} = t_{max} - t_{min} = 72 - 30 = 42 \Rightarrow B = 2 \times 42 = 84$

$C = t_{max} - \frac{B}{4} = 72 - 21 = 51$

$(\overset{\text{or}}{=} t_{min} + \frac{B}{4} = 30 + 21 = 51)$

$$\boxed{f(t) = 58\sin\left[\frac{2\pi}{84}(t-51)\right] + 62}$$

(b) Within the first 100 hours after the landing, find all the time t when the temperature is 80°F.

Solve $f(t) = 80 \Rightarrow t = \frac{84}{2\pi}\sin^{-1}\left(\frac{80-62}{58}\right) + 51$

principal sol'n: $t_p \approx \boxed{56.20}$

symmetry sol'n: $t_s = 2C + \frac{B}{2} - t_p = 2 \times 51 + 42 - 56.20$

$\approx \boxed{87.80}$

$t_p + B = 56.20 + 84 > 100$ ×

$t_p - B = 56.20 - 84 < 0$ ×

$t_s + B = 87.80 + 84 > 100$ ×

$t_s - B = 87.80 - 84 = \boxed{3.80}$ ✓