120 Exam 2 solution

Winter 2022 Su

MATH 120 Midterm Exam 2

Thursday February 24, 50 minutes during your quiz section:
AA: 8:30-9:20am @SMI 405; AB: 9:30-10:20am @SMI 405; AD: 9:30-10:20am @SIG 227
BA: 10:30-11:20am @SMI 405; BB: TuTh 11:30am-12:20pm @SMI 405

Name: | Quiz section: |

7-digit UW ID: |

Exam Instruction:

- You have 50 minutes to complete the four questions. Distribute your time wisely.
- Show your work to earn full credit. If you can not completely solve a problem, providing reasonable work and steps may still earn you some partial credit.
- Box $\boxed{\text{your final answer}}$ for each problem.
- The last page is a scratch paper, tear it off and do NOT turn it in unless you have written down additional work on it to be graded.
- **Do NOT write within 1 cm of the edge!** Your exam will be scanned for grading. If you run out of space, specify "see scratch paper", then write you additional work on the scratch paper and turn it in together with your exam.
- You can prepare one hand-written double-sided 8.5" × 11" page of notes and bring it to the exam.
- You may use a basic calculator that can NOT graph or solve equation (e.g., TI-30X IIS). All other electronic devices (e.g., cell phone, earbuds) are not allowed, and should be turned off and put away for the duration of the exam.
- You must finish the exam independently. **Giving or receiving any assistance on the exam is considered cheating, which will result in a grade of zero for the exam.** There are multiple versions of the exam, do not discuss the exam questions with other students on the exam day.

1. (12 pts) There are 25 raccoons (trash pandas) and 6 trash cans in a neighborhood now. While the raccoon population decreases by 15% every year, the number of trash cans in this neighborhood doubles every 5 years. How many years later will there be three times as many trash cans as raccoons (so that each trash panda can own three trash cans)? Leave your answer in an exact form, do *not* use calculator to round it to a decimal number.

raccoon : $P_A(t) = 25(1-0.15)^t$

$= 25(0.85)^t$

trash can: $P_B(t) = 6(2^{\frac{1}{5}})^t$

$(2P_0 = P_0 b^5 \Rightarrow b = 2^{\frac{1}{5}})$

Solve t from: $3P_A(t) = P_B(t)$

$$3 \times 25(0.85)^t = 6(2^{\frac{1}{5}})^t$$

$$\frac{75}{6} = \frac{(2^{\frac{1}{2}})^t}{(0.85)^t}$$

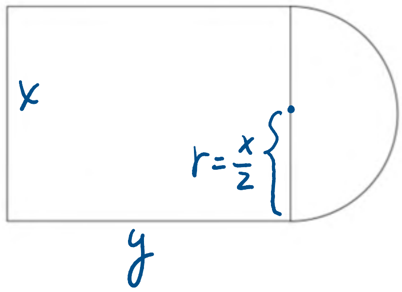
$$\frac{75}{6} = \left(\frac{2^{\frac{1}{2}}}{0.85}\right)^t$$

$$\ln\left(\frac{75}{6}\right) = t\ln\left(\frac{2^{\frac{1}{2}}}{0.85}\right)$$

$$\Rightarrow \boxed{t = \frac{\ln\left(\frac{75}{6}\right)}{\ln\left(\frac{2^{\frac{1}{2}}}{0.85}\right)}}$$

2. (15 pts) As illustrated in the picture below, a fenced enclosure consists of a semi–circle attached to a rectangle. If there is 200 meters of fencing materials available. What should the diameter of the semi–circle be in order to maximize the total fenced area? Leave your answer in an exact form, do *not* use calculator to round it to a decimal number.

(Note that the line of diameter of the semi–circle *is* part of the fence. The total fenced area includes both the half disk region and the rectangular region.)



Constraint:

$$\underbrace{\pi\left(\frac{x}{2}\right)}_{\text{perimeter of semi-circle}} + \underbrace{2x + 2y}_{\text{perimeter of rectangle}} = 200$$

$$\Rightarrow \quad y = \frac{200 - \left(\frac{\pi}{2} + 2\right)x}{2}$$

$$= 100 - \left(\frac{\pi}{4} + 1\right)x$$

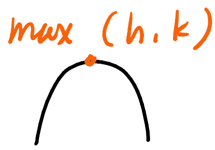
maximize area $= \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 + xy$

$$A(x) = \frac{\pi}{8}x^2 + x\left(100 - \left(\frac{\pi}{4}+1\right)x\right)$$

$$= \frac{\pi}{8}x^2 + 100x - \left(\frac{\pi}{4}+1\right)x^2$$

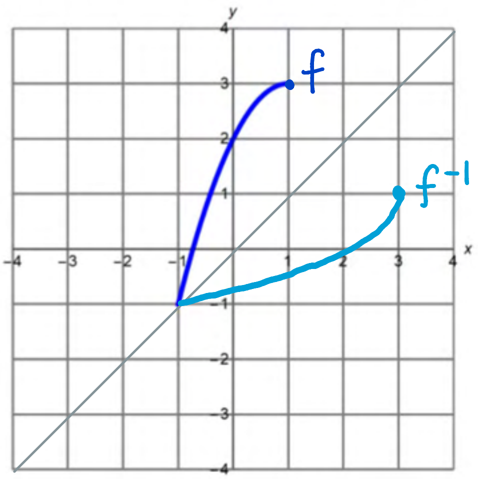
$$= \left(\frac{\pi}{8} - \frac{\pi}{4} - 1\right)x^2 + 100x$$

$$= -\left(\frac{\pi}{8}+1\right)x^2 + 100x$$



max @ $x = h = \dfrac{-100}{-2\left(\frac{\pi}{8}+1\right)} = \dfrac{50}{\frac{\pi+8}{8}} = \boxed{\dfrac{400}{\pi+8}}$

3. (12 pts) Given the graph of $f(x) = -(x-1)^2 + 3$ on a restricted domain.



(a) (2 pts) Based on the given graph of f, what is the (restricted) domain of f? what is the range of f?

domain : $-1 \le x \le 1$ or $[-1, 1]$

range : $-1 \le y \le 3$ or $[-1, 3]$

(b) (3 pts) Sketch the graph of $y = f^{-1}(x)$ in the xy–plane given above.

(c) (7 pts) Find the formula of $f^{-1}(x)$ and identify the domain and range of f^{-1}.

$y = -(x-1)^2+3$ solve x :

$(x-1)^2 = 3-y$

$x-1 = \pm\sqrt{3-y}$

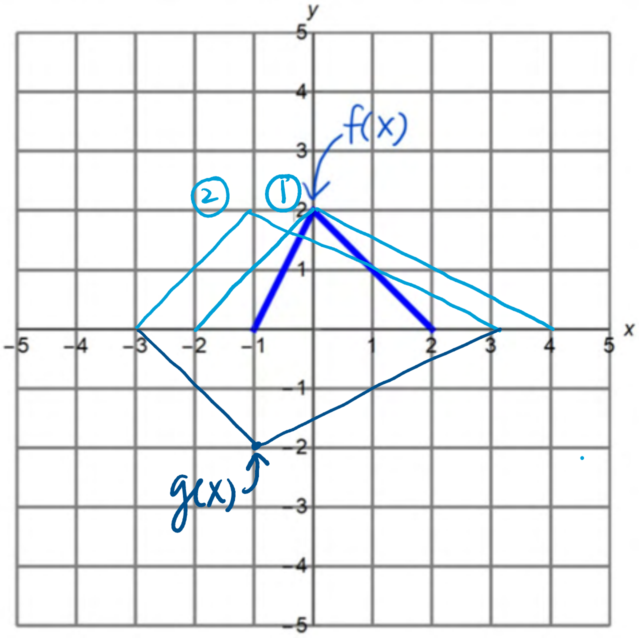
$x = 1 \pm \sqrt{3-y} \Rightarrow y = 1 \pm \sqrt{3-x} \Rightarrow f^{-1}(x) = 1 \pm \sqrt{3-x}$

range of f^{-1} = domain of f: $-1 \le y \le 1 \Rightarrow f^{-1}(x) = 1 - \sqrt{3-x}$

domain of f^{-1} = range of f: $-1 \le x \le 3$

4. (6 pts) Given the graph of $f(x)$ below. Sketch the graph of $g(x) = -f\left(\frac{x+1}{2}\right)$.

You do not need to label the intermediate graphs, only label your final graph of $g(x)$ by drawing an arrow pointing to the vertex and label $g(x)$ (similar to how $f(x)$ was labeled).



$f(x) \rightsquigarrow f(\frac{x}{2})$ ①

stretch horizontally by factor 2

$f(\frac{x}{2}) \rightsquigarrow f(\frac{x+1}{2})$ ②

shift left by 1

$f(\frac{x+1}{2}) \rightsquigarrow -f(\frac{x+1}{2}) = g(x)$

reflect across x-axis

(You will earn full credit as long as your graph of $g(x)$ is correct. If you are not confident with your graph, you may also specify the sequence of transformations being done, which may earn you some partial credit even if the graph is incorrect.)