Thursday, March 12, 2020 11:54 AM

I would recommend that you solve the problems on a sheet of paper. Then you go over the solution again and list the steps you did. Sketches help understand, so spend a sketch! The strategy is worth 4 points: 4 points for a perfect answer, 3 points if there is a minor mistake, 2 points for a more significant mistake, 1 point for a genuine, but wrong approach. 1 point is given for the correct numerical answer, no partial credit here. The whole final is worth 65 points. Here is one example of how you should present your solution.

Example The radiation level measured on Earth from a certain star is a sinusoidal function in time. At 2:30AM today, the radiation was at its maximum, 22. The level decreased to its minimum of 3 at 5:30AM today.

(a) Determine the sinusoidal function that gives the radiation level t hours after midnight.

(b) Starting from midnight, how long will it be until the level has been above 16 for a total of exactly 5 hours?

Solution

(a)
- (i) I want to find A, B, C, D in $A\sin(\frac{2\pi}{B}(x-C))+D$
- (ii) $A = \frac{maximum-minimum}{2}$.
- (iii) $D = minimum + A$.
- (iv) We can find the time between a maximum and a minimum $time_{min} - time_{max}$. This time is half the period, multiplying this time difference by 2 gives us B.
- (v) The unshifted version would have value D at midnight. So the first maximum would appear $B/4$ hours later. But our maximum appears at 2:30. The differences between the time in the unshifted version and the actual time I was given for the maximum tells me by how many units I have to shift the graph to the left or the right. This gives me C, where C is positive if it is a shift to the right and negative it is a shift to the left.
- (vi) The solution is $f(t) = 9.5\sin(\frac{2\pi}{6}(t-1)) + 12.5$.

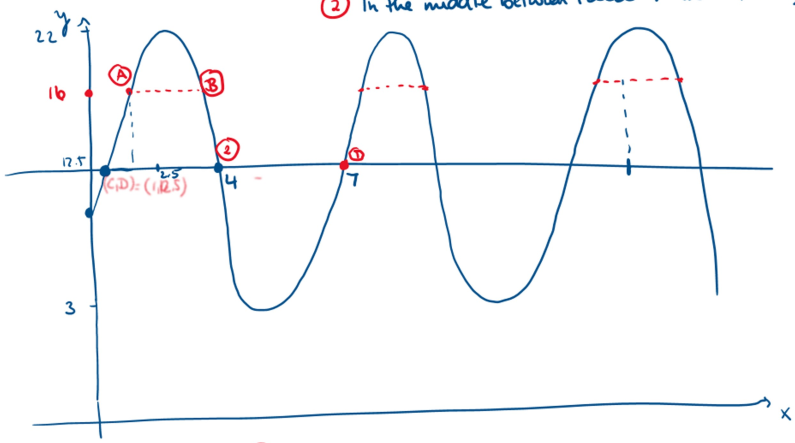
(b)
- (i) We want $f(t) > 16$. So I will find out when $f(t) = 16$ and then graphically decide which times are above and which are below 16.
- (ii) With the help of arcsin I will find the principal and symmetric solution of $f(t) = 16$.
- (iii) I will sketch a graph (see page 2) and find out which are on the rising part of the function and which are on the decreasing part of the function. The former means that right after that time the index will be above 16, the latter one that right after the time it will fall below 16.
- (iv) I will add up times where the level is above 16 until the sum is 5 hours. As soon as I reach that I get the amount of hours I have to wait until 5 hours of index higher than 16 is reached.
- (v) $t = 13.8h$

GOOD LUCK!

Example Final

Wednesday, March 11, 2020 1:12 PM

1. period 6, starting point (C,D) = (1,12.5) → next one (7,12.5)
2. In the middle between 1 and 7 at (4, 12.5)



principal solution A

symmetric solution B (argue with symmetry: distance (A,16) to (C,D) is the same as (B,16) to (4,D)

Now add up dashed ---- distances until sum is = 5, read of time when this happens.

(a) Graphical Solution/Sketch

Throughout all the problems, when you round to the fourth decimal place.

Problem 1 (5 points). Credit institution RichBank and credit institution SwissBank make two different offers on their saving account options when a deposit of $10,000 is made. RichBank offers a 6% annual interest rate, compounded monthly. SwissBank, on the other hand, offers 10% annual interest rate for the first 6 months and an annual x% for the following 6 months. What does the rate x need to be when the gain with SwissBank is $100 higher than with RichBank? Assume the accounts are kept untouched throughout. Strategy then final result, no algebra steps

1) RichBank :
   (i) monthy interest rate = $\frac{0.06}{12}$
   (ii) Amount of money after t months : $N_R(t) = 10{,}000 \cdot \left(1 + \frac{0.06}{12}\right)^t$
   (iii) Plug in $t=12$ to find amount of money after 12 months

2) Swiss Bank :
   (i) 1st half year : monthy interest rate $\frac{0.1}{12}$
   (ii) Amount of money after t months : $N_{S,1}(t) = 10{,}000 \cdot \left(1 + \frac{0.1}{12}\right)^t$
   (iii) Plug in $t=6$ to find amount after 6 months.
   (iv) 2nd half year : template now $N_{S,2nd}(t) = N_{S,1}(6) \cdot \left(1 + \frac{x}{12}\right)^t$

3) Want now : $N_R(12) + 100 = N_{S,2}(6) \cdot \left(1 + \frac{x}{12}\right)^6$

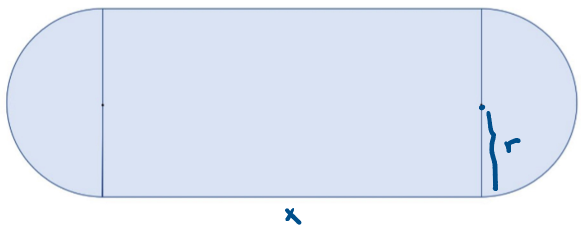
4) Solve 3) for x

5) $x \cdot 100 =$ annual rate in percentage.

Answer : 3.897%

→ Biannal compounding for SwissBank is also accepted ⇝ ~ 4%

Problem 2 (5 points)**.** You are designing the top of a pencil pouch that consists of a rectangle to which a semi circle has been attached to both ends (see the sketch). The semicircles will be made from a material that costs 1ct per square centimeter. The material of the rectangular part costs 4ct per square centimeter. The perimeter of the pouch must be 80cm. How do you have to choose the radius and the 'open' side of the rectangle so that the cost is minimal? Before you start, label your sketch! Strategy then final result, no algebra steps



(a) Sketch of Pouch

1) Find what has to minimized : Cost : Set up equation

$$C = 4 \cdot x \cdot 2r + \pi r^2$$

2) Find equation for constraint : perimeter = 80

$$80 = 2x + 2\pi r$$

3) Solve 2) for x

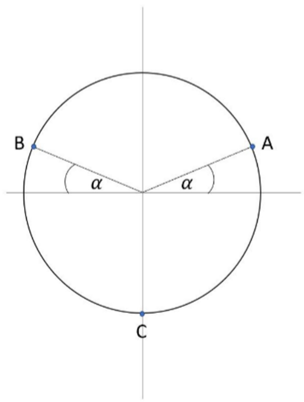
4) Plug 3) into 1) Cost becomes equation in variable r

5) Find vertex of Cost function, will be where minimum occurs

6) Take r from 5) and substitute back into 3) to find x.

solution : $r = 7.2757\text{cm}$ $x = 17.1427\text{cm}$

Problem 3 (5+5 +5 points)**.** A dog named Winston is running counterclockwise on a circular track with radius 2km. Winston's angular speed is 1.2π rad $/hr$. Winston starts at position A and it takes him $\frac{3}{4}$ hours to reach position B. Impose a coordinate system that has the center of the circular track as its origin. Units for distances should be in km.



(a) Sketch not at scale

(a) How long has Winston been running when he reaches the southern most point of the track? Strategy then final result, no algebra steps

(b) Let t be the time (in hours) that Winston has been running. Using the center of the track as the origin, express Winston's coordinates as functions of t. Strategy then final result, no algebra steps

(c) Find the distance of Winston from the point $(3, 0)$ (where the car is parked) after running for $\frac{13}{8}$ hours. Strategy then final result, no algebra steps

(a) (i) We first need to find α.

I $\omega \cdot t = \theta$

II we know ω, t and $\theta = \pi - 2\alpha$

III use I + II to solve for α

=

(ii) To reach point C, Winston has swept over $\frac{3}{2}\pi - \alpha = \theta$

(iii) $t = \frac{\theta}{\omega}$, plug θ and ω in to find t

Solution : 1.2 hours

(b) 1) We want to find r, ω, α x_0, y_0 in

$x(t) = r.\cos(\omega \cdot t + \alpha) + x_0$

$y(t) = r.\sin(\omega t + \alpha) + y_0$

$y(t) = r \cdot \sin(\omega t + \alpha) + y_0$

2) $x_0 = y_0 = 0$

3) $r = 2$ km

4) $\omega = 1.2\pi$ rad/hr

5) $\alpha = 0.05\pi$

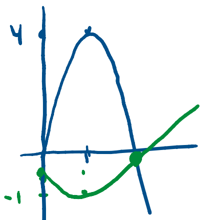
Solution : $x(t) = 2\cos(1.2\pi \cdot t + 0.05\pi)$

$y(t) = 2 \cdot \sin(1.2\pi \cdot t + 0.05\pi)$

c) 1) Plug in $t = \frac{13}{8}$ into $x(t)$ and $y(t)$ from b to find coordinates at this time

2) $d = \sqrt{\left(x\left(\frac{13}{8}\right) - 3\right)^2 + \left(y\left(\frac{13}{8}\right) - 0\right)^2}$

solution $d = 1$



not at scale

Problem 4 (5 points)**.** In a popular Washington state park, a new hiking loop is designed. We set the entrance to be the origin of a coordinate system. One leg of the new hiking loop is a parabola that starts at the entrance and has its most northern point at 1km ~~West~~ East and 4km North from the entrance. The other leg of the loop is also shaped like a parabola section. It starts 0.5 km south of the entrance and has its southern most point 1km east and 1km south from the entrance. The path ends where the legs meet (West of the entrance). Where will that be? Strategy then final result, no algebra steps 1

1) Need to find blue parabola equation:

(i) (1,4) is vertex: $y_b = a(x-1)^2+4$

(ii) (0,0) on parabola $0 = a(-1)^2+4$ solve for a

2) Need to find green parabola equation:

(i) (1,-1) is vertex $y_g = a(x-1)^2-1$

(ii) (0,-0.5) is on parabola: $-0.5 = a(0-1)^2-1$ → solve for a

3) Intersect the two parabolas and find most western point of intersection.

To do so, we set the two equations equal and solve for x.

Solution $(2.0541, -0.4444)$ West

$(-.005, -0.44)$ East

Problem 5 (5+5 points)**.** Orca J-43 is sighted in Pudget Sound at the following depths (in feet) at certain times t (in minutes) measured from sea level.

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$$D(t) = \begin{cases} 59 & 0 \le t < 5 \\ t^2 - 22t + 144 & 5 \le t < 10 \\ 24 & 10 \le t < 18 \\ 8(t-15) & 18 \le t \le 22 \end{cases}$$

(a) What is the range of the function D? Strategy then final result, no algebra steps

(b) Another orca, K-97, also shows her presence in Pudget sound. If her depths (measured at the same time as above) is given by $g(t) = 2t + 1$, at which times are both whales at the same depth? Strategy then final result, no algebra steps

a) 1) For each part of figure out the minimal and maximal functional value on the respective interval

(i) $59 = \max = \min$ throughout

(ii) Put into vertex form (at $t=22$); it is opened up, so on $5 \le t < 10$ decreasing.
Find $D(5)$, $D(10)$ to get max/min

(iii) $24 = \max = \min$

(iv) increasing line. Plug in 18, 22 to find min/max respectively.

(v) Make sure each part connects smoothly to subsequent part, so that we have no gaps

(vi) Find minimal value of all
Find maximal value of all $\Big\}$ range-interval

solution: $[24, 59]$

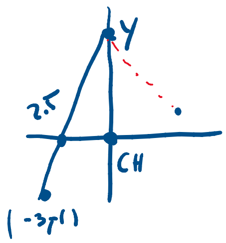
b) 1) Intersect $g = 2t+1$ and $D(t)$ by setting subsequently each part of D equal to $2t+1$. The solution t each time must be in the interval that defines the respective part, otherwise no intersection.

Solution: $t = 11.5$, $t = 20.167$

Problem 6 (5+5 points). Talia and Dina are planning to meet at he YMCA, which lies 5km otherwise no intersection.

Solution: $t = 11.5$, $t = 20.167$

Problem 6 (5+5 points)**.** Talia and Dina are planning to meet at he YMCA, which lies 5km North of the City Hall. At 3pm, Talia heads in a straight line with constant speed toward the YMCA from a point 1km South and 3km West of the City Hall. After 5 minutes she is 2.5km West, 0km South of the City Hall.



(a) Let the City Hall be the origin of a coordinate system. Find the parametric equations for Talia's route. Strategy then final result, no algebra steps

(b) Dina starts 3km East and 1km North of the City Hall on a straight line. Given that she walks at a speed of 5km per hour, at which time should she leave so that she and Talia arrive at the same time at the YMCA? Strategy then final result, no algebra steps

a) 1) Set up parametric equations $x(t) = at + b$, $y(t) = ct + d$ for Talia

by plugging in $t = 0$: $(x,y) = (-3,-1)$
$t = 5$: $(x,y) = (-2.5, 0)$

solve for a, b, c, d by combining equations

solution: $x(t) = 0.1t - 3$
$y(t) = 0.2t - 1$

b) 1) Find distance between Dina and YMCA
$d = \sqrt{(3-0)^2 + (1-5)^2}$

2) Find time she needs to go this distance: $t = \frac{d}{v}$

3) Find time Talia arrives by plugging the YMCA-coordinates into her parametric equation(s) and solve for t

4) Answer 3) + 3pm = Talia's arrival time

5) Talia's arrival time − Dina's walking time = time of departure of Dina

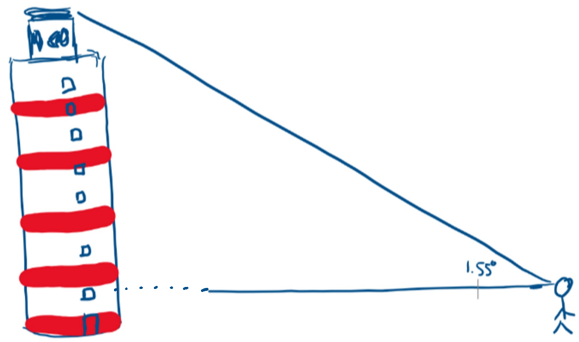
solution: Dina needs to leave at 2:30 pm

Problem 7 (5 points)**.** A person is running along a beach (assume flat ground, no incline) at a constant speed of 10km per hour. The runner notices a light house in the distance whose

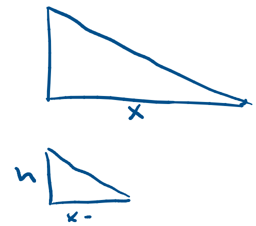
Problem 7 (5 points)**.** A person is running along a beach (assume flat ground, no incline) at a constant speed of 10km per hour. The runner notices a light house in the distance whose top shows an angle with the horizon of 1.55°. 2 minutes later, the angle is at 3.9°. What is the height of the light house when we assume that the eyes of the runner are 1.4m above the ground? Strategy then final result, no algebra steps

1

in meters



(a) Sketch not at scale



1) Find tan(1.55°) as a ratio of height and distance x between lighthouse and runner

2) Find distance runner has covered in 2 minutes. Make sure to match units

3) Set up tan(3.9) as ratio of height of lighthouse and distance x - result from 2

4) Use equation from 1, 3, combine to solve for h.

5) Add 1.4m to height h.

solution: 16.4m

Problem 8 (5+5 points)**.** The temperature in Winterberg is a sinusoidal function in time. 120 days ago, the temperature was at its maximum value of 55°F. The tempearture has been falling since then, and 20 days from today it will reach its minimum value of 10°F.

Problem 8 (5+5 points)**.** The temperature in Winterberg is a sinusoidal function in time. 120 days ago, the temperature was at its maximum value of 55°F. The tempearture has been falling since then, and 20 days from today it will reach its minimum value of 10°F.

(a) Write a function $f(t)$ for the temperature in Winterberg, in Fahrenheit, t days from today. Strategy then final result, no algebra steps 1
in standard form for sinusoidal functions

(b) People can only ski when the temperature is below 28°F. Over the next 700 days (starting today), for how many days is it cold enough to ski? Strategy then final result, no algebra steps

a) Need to determine $f(t) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$

(i) $A = (\max - \min) \cdot \frac{1}{2}$

(ii) $D = \min + A$

(iii) Find units of time between max and min
$B = 2 \cdot$ this time

(iv) The unshifted sin would have its first max at $\frac{B}{4}$

We have our first max at -120.

Take difference to determine by how many units we need to shift to the left.

We can also shift to the right: $B - |$value from here$|$

solution: $f(t) = 22.5 \cdot \sin\left(\frac{2\pi}{280}(x - 90)\right) + 32.5$

b) 1) I will first find all times when $f(t) = 28°F$

(i) Find principal solution θ,

(ii) Find symmetric solution $\pi - \theta$,

solve for t by replacing θ by $\frac{2\pi}{280}(x - 90)$

(iii) Find general solutions.

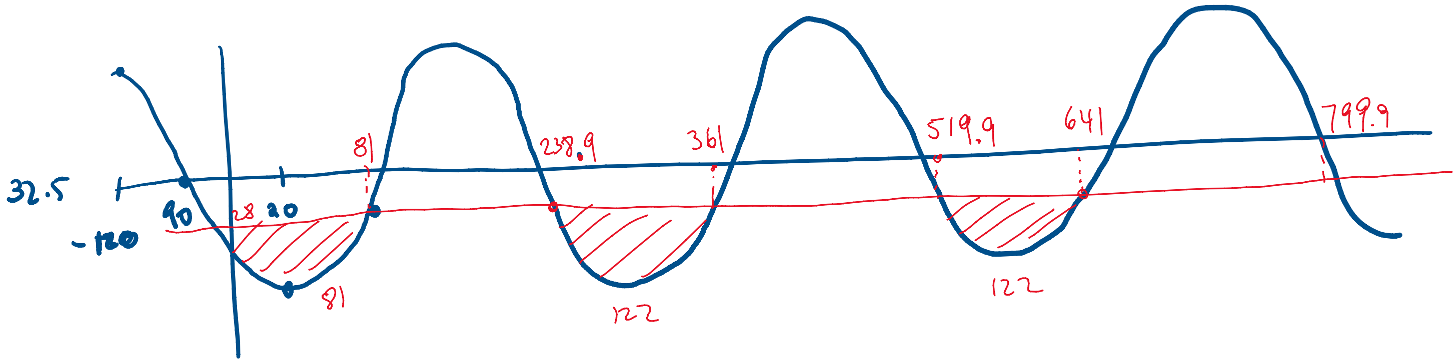
2) graph the function and determine when we are above / below 28°F.

Find time periods when we are below for the next 700 days

3) Add up those time periods

Find time periods ...

3) Add up those time periods for the next 700 days



solution : 325 days