• This exam consists of EIGHT problems on NINE pages, including this cover sheet.

• Show all work for full credit.

• You may use a scientific calculator during this exam. Graphing calculators are not permitted. Other electronic devices are not allowed, and should be turned off and put away for the duration of the exam.

• You do not need to simplify your answers.

• If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.

• If you write on the back of the page, please indicate that you have done so!

• You may use one hand-written double-sided 8.5” by 11” page of notes.

• You have 170 minutes to complete the exam.
1. **[12 points]** In the following figure (not drawn to scale), find $x$.

\[
\frac{x}{z} = \tan(40^\circ) \quad \rightarrow \quad x = z \tan(40^\circ)
\]

\[
\frac{x+y}{z} = \tan(57^\circ) \quad \rightarrow \quad x+y = z \tan(57^\circ) \quad \text{Set equal?}
\]

\[
\frac{x+y}{z+y} = \tan(50^\circ) \quad \rightarrow \quad x+y = (z+y) \tan(50^\circ)
\]

\[
z \tan(57^\circ) = z \tan(50^\circ) + 4 \tan(50^\circ)
\]

\[
z (\tan(57^\circ) - \tan(50^\circ)) = 4 \tan(50^\circ)
\]

\[
z = \frac{4 \tan(50^\circ)}{\tan(57^\circ) - \tan(50^\circ)}
\]

So...

\[
x = \frac{z \tan(40^\circ)}{4 \tan(50^\circ)} \approx 11.49
\]
2. The number of trees in Treeattle grows exponentially.

Treeattle had 600 trees in the year 2008, and 1100 trees in the year 2015.

(a) **[4 points]** Write a function \( f(x) \) for the number of trees in Treeattle, \( x \) years after the year 2000.

\[
A_0 \left( \frac{11}{6} \right)^x = 600
\]

\[
A_0 = \frac{600}{\left( \frac{11}{6} \right)^8} \approx 300.13
\]

\[
f(x) = 300.13 \left( \frac{11}{6} \right)^x
\]

(b) **[6 points]** Compute \( f^{-1}(x) \), the inverse of the function you found in part (a).

\[
x = 300.13 \left( \frac{11}{6} \right)^y
\]

\[
\frac{x}{300.13} = \left( \frac{11}{6} \right)^y
\]

\[
y = \ln \left( \frac{x}{300.13} \right)
\]

\[
y = \ln \left( \frac{300.13}{x} \right)
\]

\[
y = \ln \left( \frac{4000}{x} \right)
\]

\[
y = \frac{\ln(4000)}{\ln(1.09045)} \approx 29.9, \text{ so the year } 2030
\]

(c) **[3 points]** When will there be 4000 trees in Treeattle? Round your answer to the nearest year.
3. (a) [3 points] Write a function $f(x)$ for an upper semicircle of radius 4 centered at $(6, 2)$, defined over the interval $2 \leq x \leq 10$.

$$f(x) = 2 + \sqrt{16 - (x - 6)^2}$$

(b) [3 points] Write a function $g(x)$ for the curve obtained by taking $f(x)$ from part (a), moving it 2 units to the left, and then scaling it horizontally by a factor of $1/2$.

$$g(x) = 2 + \sqrt{16 - (2x - 4)^2}$$

(c) [4 points] Find the domain and range of $g(x)$.

(d) [3 points] Is $g(x)$ one-to-one? Explain, briefly.

No, it fails the horizontal line test!
4. [9 points] In the following configuration, wheels A and B are connected by a belt, as are wheels C and D. Wheels B and C are connected by an axle.

Wheel A has a radius of 7 feet and rotates at a speed of 6 revolutions per minute. Wheel B has a radius of 4 feet, Wheel C has a radius of 8 feet, and Wheel D has a radius of 3 feet.

How many seconds does it take Wheel D to make a complete rotation?

<table>
<thead>
<tr>
<th>Wheel</th>
<th>( \text{rpm} )</th>
<th>( \text{rad/sec} )</th>
<th>( \text{ft/sec} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \frac{7\pi}{5} )</td>
<td>( \frac{\pi}{5} )</td>
<td>( \frac{7}{r} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{7\pi}{5} )</td>
<td>( \frac{7\pi}{20} )</td>
<td>( 4 )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{14\pi}{5} )</td>
<td>( \frac{7\pi}{20} )</td>
<td>( 6 )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{14\pi}{5} )</td>
<td>( \frac{14\pi}{15} )</td>
<td>( 3 )</td>
</tr>
</tbody>
</table>

For Wheel D: \( \omega = \frac{14\pi}{15} \text{ rad/sec} \)

\[ \Theta = \omega t \]

\[ 2\pi = \frac{14\pi}{15} t \]

\[ t = \frac{30}{14} \approx 2.143 \text{ seconds} \]
5. Tori and Harry are both running **counter-clockwise** around a circular track of radius 10 meters. Tori begins at the northernmost point and Harry begins at the easternmost point. Harry runs faster.

   (a) **[4 points]** Tori first reaches the southernmost point after 8 seconds.
   
   What is Tori’s speed, in meters per second?
   
   \[
   \omega = \frac{\pi}{8} \text{ rad/sec} \\
   v = \omega r = \frac{\pi}{8} \cdot 10 = \frac{5\pi}{4} \approx 3.93 \text{ m/s}
   \]

   ![Diagram of Tori and Harry running](image)

   (b) **[6 points]** Harry begins running at the same time as Tori, and catches up to her in 11 seconds.

   What is Harry’s speed, in meters per second?

   Tori has a head start of \(\frac{\pi}{2}\) rad, so Harry runs \(\frac{\pi}{2}\) rad more than her in 11 seconds. Tori runs \((\frac{\pi}{2})11\) radians, so Harry runs \((\frac{\pi}{2})11 + \frac{\pi}{2} = \frac{15\pi}{8}\) radians in 11 seconds. His \(\omega\) is \(\frac{\frac{15\pi}{8}}{11}\) rad/sec, and so:

   \[
   v = \omega r = \frac{15\pi}{8} \cdot 10 \approx 5.355 \text{ m/s}
   \]

   (c) **[5 points]** Impose a coordinate system with units in meters and the origin at the center of the circle. After 80 seconds, what are Harry’s coordinates?

   \[
   x = r \cos(\theta + \omega t) + x_0 \\
   y = r \sin(\theta + \omega t) + y_0
   \]

   \[
   r = 10 \\
   \theta = 0 \\
   \omega = \frac{15\pi}{88}
   \]

   \[
   t = 80 \\
   x_0 = 0 \\
   y_0 = 0
   \]

   \[
   x = 10 \cos\left(\frac{15\pi}{88} \cdot 80\right) \approx 4.154 \\
   y = 10 \sin\left(\frac{15\pi}{88} \cdot 80\right) \approx -9.096
   \]
6. Consider the following multipart function:

\[ f(x) = \begin{cases} 
  x^2 + 6x + 8 & \text{if } -4 \leq x < -1 \\
  3\sin\left(\frac{2\pi}{5}(x+1)\right) + 4 & \text{if } -1 \leq x < 9 
\end{cases} \]

(a) [6 points] Sketch a graph of \( f(x) \). Label your graph clearly.

(b) [7 points] Find all solutions to the equation \( f(x) = 2 \).

First piece:
\[
2 = x^2 + 6x + 8 \\
0 = x^2 + 6x + 6 \\
x = \frac{-6 \pm \sqrt{36-24}}{2} \\
x = \frac{-6 \pm 12}{2} \quad \text{(not in domain)} \\
x = -6 \quad -4 \leq x < -1
\]

Second piece:
\[
2 = 3\sin\left(\frac{2\pi}{5}(x+1)\right) + 4 \\
-2 = 3\sin\left(\frac{2\pi}{5}(x+1)\right) \\
\frac{-2}{3} = \sin\left(\frac{2\pi}{5}(x+1)\right) \\
\sin^{-1}\left(\frac{-2}{3}\right) = \frac{2\pi}{5}(x+1) \\
x = \frac{5}{2}\sin^{-1}\left(\frac{-2}{3}\right) - 1 \approx -1.58069
\]

List of solutions:
\[
\begin{align*}
\{ & -1.58069, 2.08069, 3.41931, 7.08069, 8.41931 \} \\
& \text{in domain}
\end{align*}
\]

So overall:
\[
\begin{align*}
\{ & -1.58069, 2.08069, 3.41931, 7.08069, 8.41931 \} \\
& \text{and}
\end{align*}
\]
7. Chloë and Joë are walking around the coordinate plane. They both begin walking at the same time, in straight lines at constant speeds.

(a) [3 points] Chloë starts at \((-2, -3)\) and walks east at a speed of 4 units per second.

Give parametric equations for Chloë’s coördinates after \(t\) seconds.

\[
\begin{align*}
x &= -2 + 4t \\
y &= -3
\end{align*}
\]

(b) [4 points] Joë begins at the point \((6, 3)\) and walks towards the point \((14, -5)\), reaching it in 4 seconds.

Give parametric equations for Joë’s coördinates after \(t\) seconds.

\[
\begin{align*}
x_0 &= 6 \\
y_0 &= 3 \\
x_1 &= 14 \\
y_1 &= -5 \\
\Delta x &= 8 \\
\Delta y &= -8 \\
\Delta t &= 4 \\
x &= 6 + \frac{8}{4}t \\
y &= 3 - \frac{8}{4}t \quad \boxed{x = 6 + 2t} \\
y &= 3 - 2t \quad \boxed{y = 3 - 2t}
\end{align*}
\]

(c) [5 points] When are Chloë and Joë closest together?

\[
dist = \sqrt{(-2+4t)-(6+2t)^2 + (-3-2t)^2}
\]

\[
= \sqrt{(-8+2t)^2 + (-6+2t)^2}
\]

\[
= \sqrt{64 - 32t + 4t^2 + 36 - 24t + 4t^2}
\]

\[
= \sqrt{8t^2 - 56t + 100}
\]

\[\text{min at } t = \frac{-b}{2a} = \frac{-56}{2 \times 8} = \frac{1}{2} = 3.5 \text{ seconds}\]
8. Let \( f(x) \) be the linear-to-linear rational function with an \( x \)-intercept of 5 and a \( y \)-intercept of -4, passing through the point \((35, -6)\).

(a) [7 points] Write a formula for \( f(x) \).

\[
\begin{align*}
\frac{ax+b}{x+d} &= f(x) \\
\frac{b}{d} &= \frac{-4}{-4} = \frac{b}{d} = \frac{4}{d} \quad \Rightarrow \quad 4d = 5a, \; 5 \Rightarrow \; a = \frac{4}{5}d \\
f(0) &= 0 \quad \Rightarrow \quad \frac{5a+b}{5+d} = 0 \quad \Rightarrow \quad b = -5a \\
f(35) &= -6 \quad \Rightarrow \quad \frac{35a+b}{35+d} = -6 \quad \Rightarrow \quad 35a+b = -210-6d
\end{align*}
\]

\[
\begin{align*}
35/\left(\frac{4}{5}d\right)^{-1} &\Rightarrow 210-6d \\
d &= -7 \quad \Rightarrow \quad a = \frac{4}{5}(-7) = -5.6 \quad \Rightarrow \quad b = -4(-7) = 28
\end{align*}
\]

\[
\begin{align*}
f(x) &= \frac{-5.6x+28}{x-7}
\end{align*}
\]

(b) [2 points] Write the domain and range of \( f(x) \).

**Domain:** Everything but the vertical asymptote: \((-\infty, 7) \cup (7, \infty)\)

**Range:** Everything but the horizontal asymptote: \((-\infty, -5.6) \cup (-5.6, \infty)\)

(c) [4 points] Solve the equation \( f(f(x)) = 2 \).

\[
\begin{align*}
f(f(x)) &= 2 \\
f\left(\frac{-5.6x+28}{x-7}\right) &= 2 \\
-5.6\left(\frac{-5.6x+28}{x-7}\right)+28 &= 2 \quad \Rightarrow \quad \text{multi, nom, } \frac{b}{d}(x-7)
\end{align*}
\]

\[
\begin{align*}
\frac{-5.6(-5.6x+28)+28(x-7)}{-5.6x+28}-7(x-7) &= 2 \\
\frac{-31.36x+156.8+28x-196}{-12.6x+77} &= 2
\end{align*}
\]

\[
\begin{align*}
59.36x-352.8 &= -25.2x+154 \\
84.56x &= 506.8 \\
x &= \frac{506.8}{84.56} \approx 5.9934
\end{align*}
\]