Math 120 - Winter 2015
Final Exam
March 14, 2015

Name: ______________________ Student ID no. : ______________

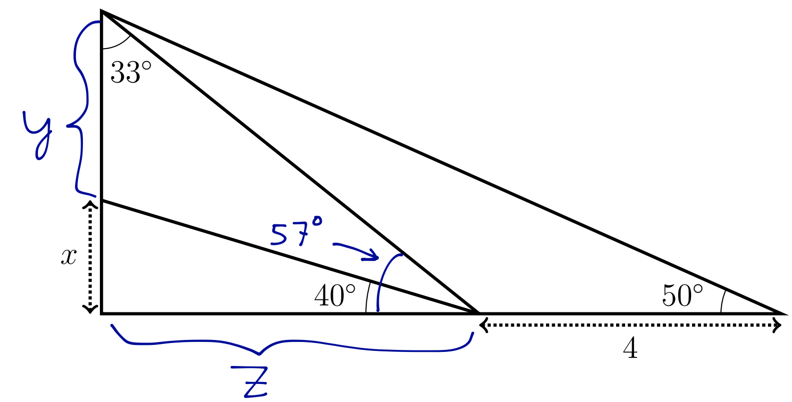
Signature: ______________________ Section: ______________

1	12	
2	13	
3	13	
4	9	
5	15	
6	13	
7	12	
8	13	
Total	100	

- This exam consists of EIGHT problems on NINE pages, including this cover sheet.
- Show all work for full credit.
- You may use a scientific calculator during this exam. Graphing calculators are not permitted. Other electronic devices are not allowed, and should be turned off and put away for the duration of the exam.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 170 minutes to complete the exam.

◦

1. **[12 points]** In the following figure (not drawn to scale), find x.



$\frac{x}{z} = \tan(40°) \longrightarrow x = z\tan(40°)$

$\frac{x+y}{z} = \tan(57°) \longrightarrow x+y = z\tan(57°)$ → set equal?

$\frac{x+y}{z+4} = \tan(50°) \longrightarrow x+y = (z+4)\tan(50°)$

$z\tan(57°) = z\tan(50°) + 4\tan(50°)$

$z(\tan(57°) - \tan(50°)) = 4\tan(50°)$

$z = \frac{4\tan(50°)}{\tan(57°) - \tan(50°)}$

So.... $x = z\tan(40°)$

$$\boxed{x = \left(\frac{4\tan(50°)}{\tan(57°) - \tan(50°)}\right)\tan(40°) \approx 11.49}$$

2. The number of trees in Treeattle grows exponentially.

Treeattle had 600 trees in the year 2008, and 1100 trees in the year 2015.

(a) **[4 points]** Write a function $f(x)$ for the number of trees in Treeattle, x years after the year 2000.

$$f(x) = A_0 b^x$$

$$f(8) = 600 = A_0 b^8$$

$$f(15) = 1100 = A_0 b^{15}$$

$\downarrow$ Divide

$$b^7 = \frac{11}{6}$$

$$b = \left(\frac{11}{6}\right)^{1/7} \approx 1.09045$$

$$A_0\left(\left(\frac{11}{6}\right)^{1/7}\right)^8 = 600$$

$$A_0 = \frac{600}{\left(\frac{11}{6}\right)^{8/7}} \approx 300.13$$

$$\boxed{f(x) = 300.13(1.09045)^x}$$

(b) **[6 points]** Compute $f^{-1}(x)$, the inverse of the function you found in part (a).

$$x = 300.13(1.09045)^y$$

$$\frac{x}{300.13} = (1.09045)^y$$

$$\ln\left(\frac{x}{300.13}\right) = y \ln(1.09045)$$

$$y = \boxed{f^{-1}(x) = \frac{\ln\left(\frac{x}{300.13}\right)}{\ln(1.09045)}}$$

(c) **[3 points]** When will there be 4000 trees in Treeattle? Round your answer to the nearest year.

$$f^{-1}(4000) = \frac{\ln\left(\frac{4000}{300.13}\right)}{\ln(1.09045)} \approx 29.9, \text{ so the year } \boxed{2030}$$

3. (a) **[3 points]** Write a function $f(x)$ for an upper semicircle of radius 4 centered at $(6,2)$, defined over the interval $2 \le x \le 10$.

$$\boxed{f(x) = 2 + \sqrt{16 - (x-6)^2}}$$

(b) **[3 points]** Write a function $g(x)$ for the curve obtained by taking $f(x)$ from part (a), moving it 2 units to the left, and *then* scaling it horizontally by a factor of $1/2$.

$$y = 2 + \sqrt{16-(x-6)^2}$$

left 2:

$$y = 2 + \sqrt{16 - ((x+2)-6)^2}$$

Scale horiz. by ½:

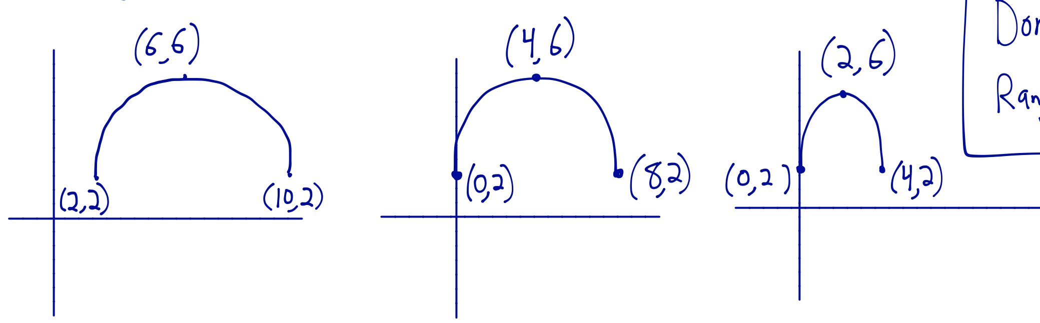
$$y = 2 + \sqrt{16 - \left(\left(\left(\tfrac{x}{1/2}\right)+2\right)-6\right)^2}$$

$$g(x) = 2 + \sqrt{16 - (2x-4)^2}$$

$$\boxed{g(x) = 2 + \sqrt{16 - (2x-4)^2}}$$

(c) **[4 points]** Find the domain and range of $g(x)$.

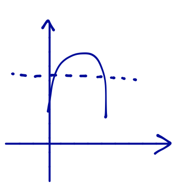
Original: Shift: Scale:



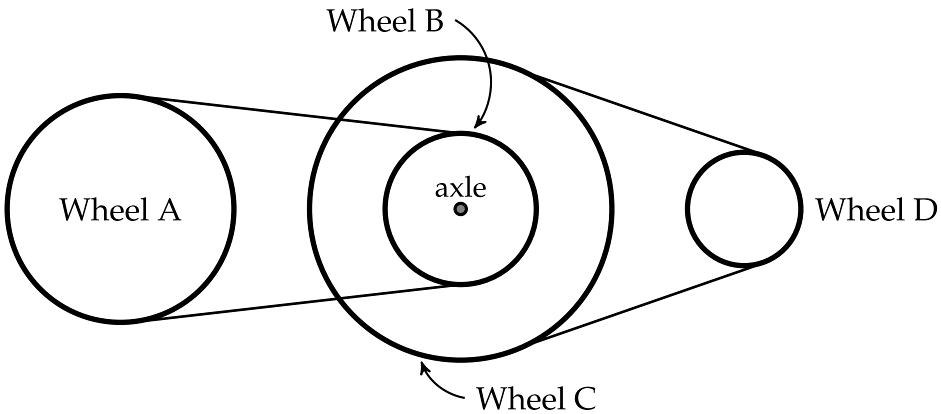
Domain: $[0,4]$
Range: $[2,6]$

(d) **[3 points]** Is $g(x)$ one-to-one? Explain, briefly.

$\boxed{\text{No,}}$ it fails the horizontal line test!



4. **[9 points]** In the following configuration, wheels A and B are connected by a belt, as are wheels C and D. Wheels B and C are connected by an axle.



Wheel A has a radius of 7 feet and rotates at a speed of 6 revolutions per minute.

Wheel B has a radius of 4 feet, Wheel C has a radius of 8 feet, and Wheel D has a radius of 3 feet.

How many seconds does it take Wheel D to make a complete rotation?

$$\omega = \frac{12\pi}{60} \text{ rad/sec} = \frac{\pi}{5} \text{ rad/sec}$$

Wheel	(ft/sec) v	= (rad/sec) ω	× (ft) r
A	$\frac{7\pi}{5}$	$\frac{\pi}{5}$	7
B	$\frac{7\pi}{5}$	$\frac{7\pi}{20}$	4
C	$\frac{14\pi}{5}$	$\frac{7\pi}{20}$	8
D	$\frac{14\pi}{5}$	$\frac{14\pi}{15}$	3

For wheel D: $\omega = \frac{14\pi}{15}$ rad/sec

$\theta = \omega t$, so $2\pi = \frac{14}{15}\pi t$

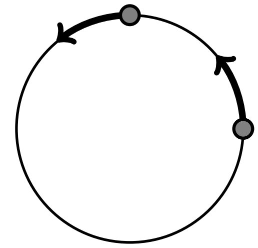
$\Downarrow$

$$\boxed{t = \frac{30}{14} \approx 2.143 \text{ seconds}}$$

5. Tori and Harry are both running **counter-clockwise** around a circular track of radius 10 meters. Tori begins at the northernmost point and Harry begins at the easternmost point. Harry runs faster.

(a) **[4 points]** Tori first reaches the southernmost point after 8 seconds.

What is Tori's speed, in meters per second?



$\omega = \frac{\pi}{8}$ rad/sec

$v = \omega r = \frac{\pi}{8} \cdot 10 = \frac{5\pi}{4} \approx \boxed{3.93 \text{ m/s}}$

(b) **[6 points]** Harry begins running at the same time as Tori, and catches up to her in 11 seconds.

What is Harry's speed, in meters per second?

Tori has a head start of $\pi/2$ rad, so Harry runs $\frac{\pi}{2}$ rad more than her in 11 seconds. Tori runs $(\frac{\pi}{8})(11)$ radians, so Harry runs $(\frac{\pi}{8} \cdot 11) + \frac{\pi}{2} = \frac{15\pi}{8}$ radians in 11 seconds. His ω is $\frac{\frac{15\pi}{8}}{11} = \frac{15\pi}{88}$ rad/sec, and so:

$v = \omega r = \frac{15\pi}{88} \cdot 10 \approx \boxed{5.355 \text{ m/s}}$

(c) **[5 points]** Impose a coordinate system with units in meters and the origin at the center of the circle. After 80 seconds, what are Harry's coordinates?

$x = r\cos(\theta_0 + \omega t) + x_0$

$y = r\sin(\theta_0 + \omega t) + y_0$

$r = 10$, $\theta_0 = 0$, $\omega = \frac{15\pi}{88}$, $t = 80$, $x_0 = 0$, $y_0 = 0$

$x = 10\cos\left(\frac{15\pi}{88} \cdot 80\right) \approx \boxed{4.154}$

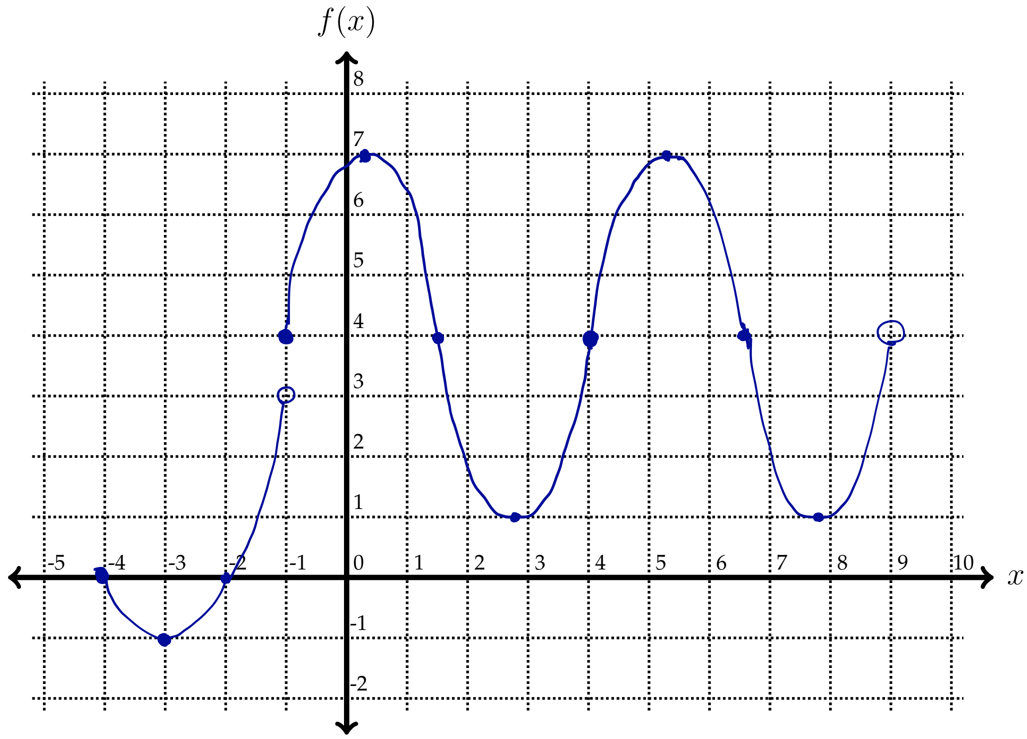
$y = 10\sin\left(\frac{15\pi}{88} \cdot 80\right) \approx \boxed{-9.096}$

6. Consider the following multipart function:

$\to (x+3)^2 - 1$

$$f(x) = \begin{cases} x^2 + 6x + 8 & \text{if } -4 \le x < -1 \\ 3\sin\left(\frac{2\pi}{5}(x+1)\right) + 4 & \text{if } -1 \le x < 9 \end{cases}$$

(a) **[6 points]** Sketch a graph of $f(x)$. Label your graph clearly.



(b) **[7 points]** Find *all solutions* to the equation $f(x) = 2$.

First piece:

$2 = x^2 + 6x + 8$

$0 = x^2 + 6x + 6$

$x = \frac{-6 \pm \sqrt{36-24}}{2}$

$x = \frac{-6+\sqrt{12}}{2}$ } yup

$\underbrace{x = \frac{-6-\sqrt{12}}{2}}_{\text{not in domain } -4 \le x < -1}$

Second piece:

$2 = 3\sin\left(\frac{2\pi}{5}(x+1)\right) + 4$

$\frac{-2}{3} = \sin\left(\frac{2\pi}{5}(x+1)\right)$

$\sin^{-1}\left(\frac{-2}{3}\right) = \frac{2\pi}{5}(x+1)$

$x = \frac{5}{2\pi}\left(\sin^{-1}\left(\frac{-2}{3}\right)\right) - 1 \approx \overbrace{-1.58069}^{\text{principal sol'n}}$

Symmetry solution

$= 2C + \frac{B}{2} - p = 2.08069$

List of solutions:

-1.58069

2.08069
3.41931
7.08069
8.41931 } In domain $-1 \le x < 9$

12.08069

$\vdots$

$\boxed{\text{So overall: } \frac{-6+\sqrt{12}}{2},\ 2.08069,\ 3.41931,\ 7.08069, \text{ and } 8.41931}$

7. Chloë and and Joë are walking around the coördinate plane. They both begin walking at the same time, in straight lines at constant speeds.

   (a) **[3 points]** Chloë starts at $(-2,-3)$ and walks east at a speed of 4 units per second.

   Give parametric equations for Chloë's coördinates after t seconds.

$$\boxed{\begin{aligned} x &= -2 + 4t \\ y &= -3 \end{aligned}}$$

   (b) **[4 points]** Joë begins at the point $(6,3)$ and walks towards the point $(14,-5)$, reaching it in 4 seconds.

   Give parametric equations for Joë's coördinates after t seconds.

$x_0 = 6$ $\quad y_0 = 3$

$x_1 = 14$ $\quad y_1 = -5$

$\Delta x = 8$ $\quad \Delta y = -8$

$\Delta t = 4$

$x = 6 + \frac{8}{4}t$

$y = 3 + \frac{-8}{4}t$

$$\boxed{\begin{aligned} x &= 6 + 2t \\ y &= 3 - 2t \end{aligned}}$$

   (c) **[5 points]** When are Chloë and Joë closest together?

$$\text{dist} = \sqrt{\left((-2+4t)-(6+2t)\right)^2 + \left(-3-(3-2t)\right)^2}$$

$$= \sqrt{(-8+2t)^2 + (-6+2t)^2}$$

$$= \sqrt{64 - 32t + 4t^2 + 36 - 24t + 4t^2}$$

$$= \sqrt{8t^2 - 56t + 100} \leftarrow \text{quadratic!}$$

min at $h = \frac{-b}{2a} = \frac{56}{16} = \boxed{3.5 \text{ seconds}}$

8. Let $f(x)$ be the linear-to-linear rational function with an x-intercept of 5 and a y-intercept of -4, passing through the point $(35, -6)$.

(a) **[7 points]** Write a formula for $f(x)$.

$f(x) = \frac{ax+b}{x+d}$

$f(0) = -4 \rightarrow \frac{b}{d} = -4 \rightarrow b = -4d \rightarrow -4d = -5a$, so $a = \frac{4}{5}d$

$f(5) = 0 \rightarrow \frac{5a+b}{5+d} = 0 \rightarrow b = -5a$

$f(35) = -6 \rightarrow \frac{35a+b}{35+d} = -6 \rightarrow 35a + b = -210 - 6d$

$35\left(\frac{4}{5}d\right) + (-4d) = -210 - 6d$

$30d = -210$

$d = -7$

$a = \frac{4}{5}(-7) = -5.6$

$b = -4(-7) = 28$

$\boxed{f(x) = \frac{-5.6x+28}{x-7}}$

(b) **[2 points]** Write the domain and range of $f(x)$.

Domain: Everything but the vertical asymptote: $(-\infty, 7) \cup (7, \infty)$

Range: Everything but the horizontal asymptote: $(-\infty, -5.6) \cup (-5.6, \infty)$

(c) **[4 points]** Solve the equation $f(f(x)) = 2$.

$f(f(x)) = 2$

$f\left(\frac{-5.6x+28}{x-7}\right) = 2$

$\frac{-5.6\left(\frac{-5.6x+28}{x-7}\right) + 28}{\left(\frac{-5.6x+28}{x-7}\right) - 7} = 2$

mult. num. & denom. by $(x-7)$

$\frac{-5.6(-5.6x+28) + 28(x-7)}{(-5.6x+28) - 7(x-7)} = 2$

$\frac{31.36x - 156.8 + 28x - 196}{-12.6x + 77} = 2$

$59.36x - 352.8 = -25.2x + 154$

$84.56x = 506.8$

$x = \frac{506.8}{84.56} \approx \boxed{5.9934}$