Math 120AB Exam I - Winter 05 - Schneider

Version one - Solutions.

Problem 1. Suppose that

$$f(x) = -2x^2 - 3x + 1 \qquad g(x) = \sqrt{4x-2} \qquad h(x) = 3x$$

(a) Compute the composition $f(g(h(x)))$.

$f(g(h(x))) = f(g(3x)) = f(\sqrt{12x-2}) = -2(12x-2) - 3\sqrt{12x-2} + 1 = -24x - 3\sqrt{12x-2} + 5$

(b) Compute the composition $g(g(x))$.

$$g(g(x)) = g(\sqrt{4x-2}) = \sqrt{4\sqrt{4x-2} - 2}$$

(c) Simplify

$$\frac{f(x+h) - f(x)}{h}$$

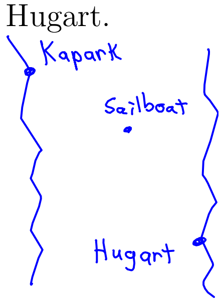
to the point where setting $h = 0$ is allowed.

$$\frac{f(x+h) - f(x)}{h} = \frac{-2(x+h)^2 - 3(x+h) + 1 - (-2x^2 - 3x + 1)}{h} =$$

$$\frac{-2x^2 - 4xh - 2h^2 - 3x - 3h + 1 + 2x^2 + 3x - 1}{h} =$$

$$\frac{-4xh - 2h^2 - 3h}{h} = -4x - 2h - 3$$

Problem 2. Johnny's sailboat is floating 6 miles west and 10 miles north of the port city of Hugart. A speed boat leaves Hugart at 2:30PM and travels in a straight line to the port city Kapark on the other side of the bay. Kapark is 10 miles west and 14 miles north of Hugart.



(a) Where is the speedboat located when it is closest to Johnny's sailboat?

You can put the origin where you like, the correct answers for the different cases are correct. Here we assume $(0,0)$ is at Hugart. The speedboat goes from $(0,0)$ to $(-10,14)$ so it has the equation $y = \frac{-7}{5}x$. The perpendicular line has slope $5/7$ and goes through the point $(-6,10)$ so its equation is

$$y = \frac{5}{7}x + \frac{100}{7}$$

Setting them equal

$$-75x = \frac{5}{7}x + \frac{100}{7}$$

and solving for x gives $x = -6.757$ and $y = 9.459$.

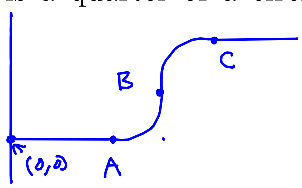
(b) Assuming the speedboat travels at 12 miles per hour, at what time is it closest to Johnny's sailboat? (Give the time in the format of hours and minutes, for example 2:30PM).

You need the distance from $Hugart$ to the point found above, that is

$$\sqrt{6.757^2 + 9.459^2} = 11.625$$

then divide by 12 MPH to get .969 hours, or 58.124 minutes, so at $3:28$PM.

Problem 3. The graph of the function $f(x)$ is given below. In the graph, the point A is $(8, 0)$, the point B is $(12, 4)$, and the point C is $(16, 8)$. Between the points A and B, $f(x)$ is a quarter of a circle. Between the points B and C, $f(x)$ is also a quarter of a circle.



(a)Find the multipart formula for $f(x)$ on the domain $x \geq 0$.

circle one has center $(8, 4)$ and radius 4.
circle two has center $(16, 4)$ and radius 4.
We want the lower part of circle one, and the upper of circle two:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 8 \\ 4 - \sqrt{16 - (x-8)^2} & \text{if } 8 \leq x \leq 12 \\ 4 + \sqrt{16 - (x-16)^2} & \text{if } 12 \leq x \leq 16 \\ 8 & \text{if } 16 \leq x \end{cases}$$

(b) Find a and b such that if $a \leq x \leq b$, then $2 \leq f(x) \leq 6$.

if $a \leq x \leq b$, makes $2 \leq f(x) \leq 6$, then it must be that $f(a) = 2$ and $f(b) = 6$. The function is increasing, so if we find the two x values (which the problem calls a and b) that have y values of 2 and 6, then all x between these values go to y values between 2 and 6.

So we solve $f(x) = 2$, we see this is on circle one, circle one has equation $(x-8)^2 + (y-4)^2 = 16$ plug in $y = 2$ and get $x = 8 \pm \sqrt{12}$ and take the solution to be $a = 8 + \sqrt{12}$, since it is the lower right quarter of the circle that makes up $f(x)$.
Then solve $f(x) = 6$, we see this is on circle two, circle two has equation $(x-16)^2 + (y-4)^2 = 16$ plug in $y = 6$ and get $x = 16 \pm \sqrt{12}$ and take the solution to be $b = 16 - \sqrt{12}$, since it is the lower left quarter of the circle that makes up $f(x)$.

Problem 4. Dr. Skandera has started a chemical reaction involving the two substances, substance A and substance B. Initially, at time $t = 0$, there are 10 grams of each. The amount of each substance in grams, is given by the following functions

$$A(t) = 10 + \frac{1}{2}t$$
$$B(t) = \frac{7}{25}(t-5)^2 + 3$$

where t is minutes since the start of the reaction.

(a) Is there a time later then 0 when the amount of substance A and substance B is the same? If so, find this time and the amount of each substance at this time.

You solve for $A(t) = B(t)$, or

$$10 + \frac{1}{2}t = \frac{7}{25}(t-5)^2 + 3$$

after correct algebra, you get

$$\frac{33}{10}t - \frac{7}{25}t^2 = 0$$

or $.33t - .28t^2 = 0$ You do not need the quadratic formula, one solution is $t = 0$, divide out by t and you are left with $3.3 - .28t = 0$ which gives the solution $t = 11.786$ minutes. The problem also asks for the quantity, plug this $t = 11.786$ into either $A(t)$ or $B(t)$, $A(t)$ is easier as it is a line, and get $A(11.786) = 15.893$ grams of each chemical.

(b) Is there a time when the difference between substance A and substance B has a maximum value? If so, find this time and this maximum value.

This part of the question is poorly worded, the intention was to find the time when $A(t) - B(t)$ is the greatest. It is reasonable to interpret the difference as $\|B(t) - A(t)\|$ as well, that is to assume the difference is something positive. In any case, full credit requires an investigation of the vertex of $A(t) - B(t)$ or $B(t) - A(t)$, and to not interpret a negative y value as a maximum. To find the vertex of $A(t) - B(t)$, the quadratic is

$$-.28t^2 + 3.3t$$

and so the max occurs at $\frac{-b}{2a} = 5.893$, minutes, and the value is $-.28 * 5.893^2 + 3.3 * 5.893 = 9.723$ grams.