Math 120AB - Winter 2004
Mid-Term Exam Number One Solutions
January 29, 2004

1. *On Canary Island, there were 22,000 canaries in the year 1935. In 1950, there were 25,400. Assume that the number of canaries is a linear function of time.*

   In what year will there be twice as many canaries as there were in 1985?

   We begin by finding the equation of the line through the points (1935, 22000) and (1950, 25400).

   The slope is

   $$m = \frac{25400 - 22000}{1950 - 1935} = \frac{680}{3} \approx 226.666667$$

   so the equation of the line is

   $$y - 22000 = 226.666667(x - 1935)$$

   i.e.,

   $$y = 226.6666667(x - 1935) + 22000.$$

   In 1985, the population was

   $$226.6666667(1985 - 1935) + 22000 \approx 33333.33333.$$

   The question is thus asking when the population will be 66666.66666:

   $$66666.66666 = 226.6666667(x - 1935) + 22000$$

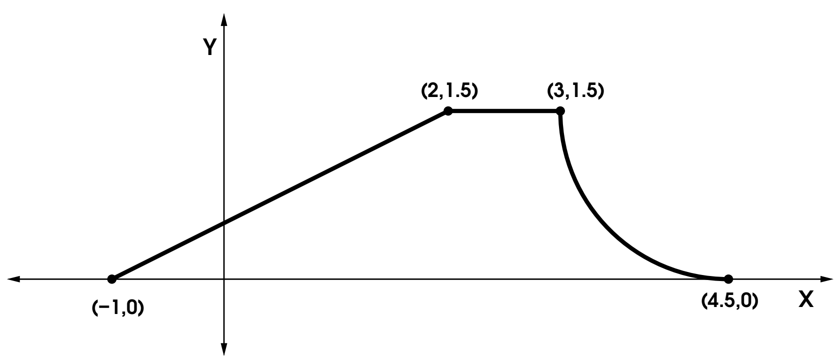
   $$44666.6666 = 226.6666667(x - 1935)$$

   $$197.0588 = x - 1935$$

   $$x = 1935 + 197.0588 = 2132.0588.$$

   So in the year 2132 there will be twice as many canaries as in the year 1985.

2. *Here is the graph of a function, $g(x)$:*



   The graph consists of two line segments and a quarter-circle arc.

(a) *Write the mulipart rule for $g(x)$.*

The sloping line on the interval $-1 \leq x \leq 2$ has equation

$$y - 0 = \frac{1.5 - 0}{2 - (-1)}(x - (-1)),$$

i.e.,

$$y = \frac{1}{2}(x + 1).$$

The circle of which the quarter-circle arc is a part has center $(4.5, 1.5)$ and radius 1.5, so it has equation

$$(x - 4.5)^2 + (y - 1.5)^2 = 1.5^2 = 2.25.$$

Solving for y we have

$$y = 1.5 \pm \sqrt{1.5^2 - (x - 4.5)^2}$$

We want the lower portion of the circle, on the interval $3 \leq x \leq 4.5$. All together, we have

$$g(x) = \begin{cases} \frac{1}{2}(x + 1) & \text{if } -1 \leq x \leq 2, \\ 1.5 & \text{if } 2 < x \leq 3, \\ 1.5 - \sqrt{1.5^2 - (x - 4.5)^2} & \text{if } 3 \leq x \leq 4.5. \end{cases}$$

(b) *Find all solutions to the equation*

$$g(x) = 0.2.$$

If $g(x) = \frac{1}{2}(x + 1) = 0.2$, then

$$x + 1 = 0.4$$

$$x = -0.6.$$

Notice that if $x = -0.6$, then $g(x) = \frac{1}{2}(x + 1) = 0.2$, so this is a solution.

If $g(x) = 1.5$, then $g(x) \neq 0.2$.

If $g(x) = 1.5 - \sqrt{1.5^2 - (x - 4.5)^2} = 0.2$ then

$$-1.3 = -\sqrt{1.5^2 - (x - 4.5)^2}$$

$$1.3^2 = 1.5^2 - (x - 4.5)^2$$

$$0.56 = (x - 4.5)^2$$

$$\sqrt{0.56} = |x - 4.5|$$

i.e.,

$$x - 4.5 = \pm\sqrt{0.56}$$

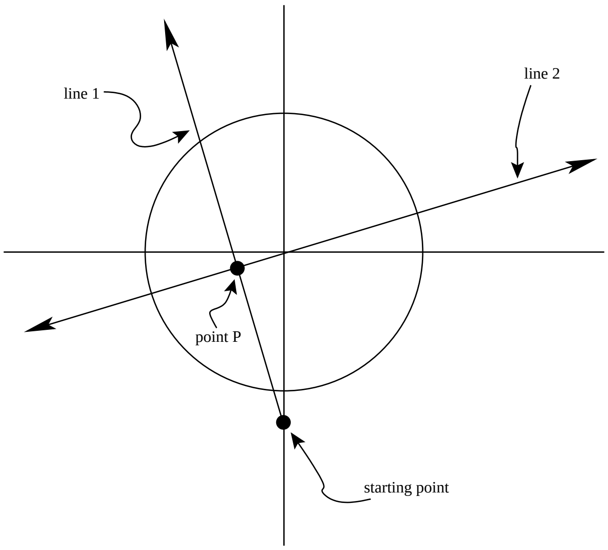
$$x = 4.5 \pm \sqrt{0.56}$$

Since $g(x) = 1.5 - \sqrt{1.5^2 - (x-4.5)^2}$ only if $3 \leq x \leq 4.5$, the only solution with this equation is $x = 4.5 - \sqrt{0.56}$. All together then, the solutions are

$$x = -0.6 \quad \text{and} \quad x = 4.5 - \sqrt{0.56} \approx 3.7516685226.$$

3. *Stan is planning to ride his mountain bike across the Circular Desert. He intends to take a straight-line route from a point 50 miles due south of the center of the desert to a point 60 miles north and 30 miles west of the center. The desert is a circular region with a radius of 40 miles.*

   How close will Stan get to the center of the desert?



We need to find the equation of line 1 (Stan's path through the desert), then find the equation of line 2 (the line perpendicular to line 1 passing through the origin), and then find the intersection of lines 1 and 2.

The equation of line 1 is:

$$y = -50 + \frac{60-(-50)}{-30-0}(x-0) = -50 - \frac{11}{3}x.$$

Line 2 has slope $\frac{3}{11}$, so its equation is

$$y = \frac{3}{11}x$$

The intersection of line 1 and line 2 (point P) is then found by solving

$$\frac{3}{11}x = -50 - \frac{11}{3}x$$

$$\frac{130}{33}x = -50$$

$$x = -\frac{165}{13}.$$

So point P is the point $\left(-\frac{165}{13}, -\frac{165}{13}\frac{3}{11}\right) = \left(-\frac{165}{13}, -\frac{45}{13}\right)$.

Thus, the closest Stan gets to the center of the desert is

$$\sqrt{\left(-\frac{165}{13}\right)^2 + \left(-\frac{45}{13}\right)^2} \approx 13.155870289605 \text{ miles.}$$

4. *Find the constant(s) d so that the graph of the quadratic function*

$$f(x) = (x + d)(x + 6d) + 1$$

has its vertex on the x-axis.

We can rewrite $f(x)$ in vertex form:

$$f(x) = x^2 + dx + 6dx + 6d^2 + 1$$

$$= x^2 + 7dx + 6d^2 + 1.$$

$$= (x + \frac{7}{2}d)^2 - \left(\frac{7}{2}d\right)^2 + 6d^2 + 1$$

$$= (x + \frac{7}{2}d)^2 - \frac{25}{4}d^2 + 1.$$

So we see the vertex of $f(x)$ is the point $\left(-\frac{7}{2}d, 1 - \frac{25}{4}d^2\right)$. So to have the vertex on the x-axis we need

$$1 - \frac{25}{4}d^2 = 0$$

i.e.,

$$d^2 = \frac{4}{25}$$

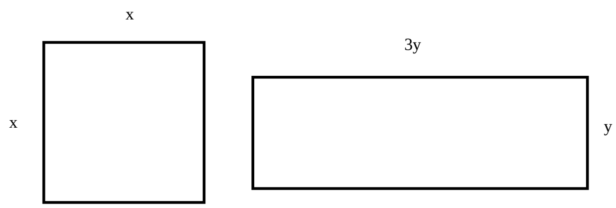
so

$$|d| = \frac{2}{5}$$

i.e.,

$$d = \pm\frac{2}{5}.$$

5. *You have 100 meters of fencing with which to make two enclosures. One enclosure will be square, and the other will be rectangular, with its long side three times the length of its short side. For example, the enclosures might look like this:*



What should the dimensions of the square be to get the least possible total combined area?

We can label the dimensions of the rectangles as shown above. Then the total perimeter of the two rectangles is $4x + 6y$. So

$$100 = 4x + 8y.$$

The total combined area of the rectangles is

$$A = x^2 + 3y^2.$$

Solving $100 = 4x + 8y$ for y we have

$$y = \frac{100 - 4x}{8}$$

so that

$$\begin{aligned} A &= x^2 + 3\left(\frac{100 - 4x}{8}\right)^2 \\ &= \frac{7}{4}x^2 - \frac{75}{2}x + \frac{1875}{4} \\ &= \frac{7}{4}\left(x^2 - \frac{150}{7}x\right) + \frac{1875}{4} \\ &= \frac{7}{4}\left(x - \frac{75}{7}\right)^2 - \frac{7}{4}\left(\frac{75}{7}\right)^2 + \frac{1875}{4} \end{aligned}$$

Since this is in vertex form, and this quadratic has positive leading coefficient, we can see that the minimum value of A is achieved when $x = \dfrac{75}{7}$.