Solutions of midterm I, 5/5/2004:

1. (a) $f_1(f_1(x)) = f_1(x) = x$, $f_3(f_2(x)) = f_3(\sqrt{4-x^2}) = 1$, $f_4(f_2(x)) = f_4(\sqrt{4-x^2}) = 4(\sqrt{4-x^2})^2 - 5 = 11 - 4x^2$, $f_4(f_3(x)) = f_4(1) = -1$.

(b) f_4 doesn't need any restrictions on the domain. So we only need valid numbers for f_2 which is $4 - x^2 \geq 0$, so we have the domain= $\{x| - 2 \leq x \leq 2\}$.

2. Given 2 points A: (0, $\frac{1}{2}$), B: (3, -1) **or** A: (3, $\frac{1}{2}$), B: (0, -1).

(a) From $|QA| = |QB|$, we get the equation

$$\sqrt{(x-0)^2 + (y-\frac{1}{2})^2} = \sqrt{(x-3)^2 + (y-(-1))^2} \textbf{ or } \sqrt{(x-3)^2 + (y-\frac{1}{2})^2} = \sqrt{(x-0)^2 + (y-(-1))^2}$$

(b) We'll simplify the equation above so that we can write y as a function of x.

$$\begin{aligned}
\sqrt{(x-0)^2 + (y-\tfrac{1}{2})^2} &= \sqrt{(x-3)^2 + (y-(-1))^2} \\
(x-0)^2 + (y-\tfrac{1}{2})^2 &= (x-3)^2 + (y-(-1))^2 \\
x^2 + y^2 - y + \tfrac{1}{4} &= x^2 - 6x + 9 + y^2 + 2y + 1 \\
-y + \tfrac{1}{4} &= -6x + 9 + 2y + 1 \\
-3y &= -6x + 9\tfrac{3}{4} \\
y &= \tfrac{1}{-3}(-6x + \tfrac{39}{4}) \\
y &= 2x - \tfrac{13}{4}
\end{aligned}$$

or

$$\begin{aligned}
\sqrt{(x-3)^2 + (y-\tfrac{1}{2})^2} &= \sqrt{(x-0)^2 + (y-(-1))^2} \\
(x-3)^2 + (y-\tfrac{1}{2})^2 &= (x-0)^2 + (y-(-1))^2 \\
x^2 - 6x + 9 + y^2 - y + \tfrac{1}{4} &= x^2 + y^2 + 2y + 1 \\
-6x + 9 - y + \tfrac{1}{4} &= 2y + 1 \\
-3y &= 6x - 9 - \tfrac{1}{4} + 1 \\
y &= \tfrac{1}{-3}(6x - 8\tfrac{1}{4}) \\
y &= -2x + \tfrac{11}{4}
\end{aligned}$$

3 (a) rate of change $= \frac{8-2}{20-0} = 0.3$ hundred employees per month.

(b) A key idea here is that to keep y with the unit "hundred" and just use the numbers on the graph. Otherwise, we **will not** have the circle equation if we start to denote y with 200 or 800 as its values. The reason is clear that changing form 2 to 200 is rescaling the graph vertically and it will stretch the circle to become an ellipse. The circle equation is $(x-27)^2 + (y-8)^2 = 7^2$ with x in month and y in hundred employees. Then we have

$$y = f(x) = \begin{cases} 0.3x + 2, & 0 \leq x \leq 20 \\ 8 + \sqrt{49 - (x-27)^2}, & 20 \leq x \leq 27 \\ 8 - \sqrt{49 - (x-27)^2}, & 27 < x \leq 34 \end{cases}$$

(c) For the similar reason in (b) the equation we need to solve is

$$\begin{aligned}
10 &= 8 + \sqrt{49 - (x-27)^2}, \quad \text{for } 20 \leq x \leq 27 \\
2^2 &= 49 - (x-27)^2 \\
45 &= (x-27)^2 \\
x &= 27 \pm \sqrt{45}
\end{aligned}$$

so we take $x = 27 - \sqrt{45}$. Then the period that the company having more than 10 hundred employees is from $= 27 - \sqrt{45}$ to 27 which lasts $\sqrt{45} \approx 6.71$ months.

4. A stone is moving along the path that follows the graph of $y = f(x) = -x^2 + 9x + 10$ **or** $y = f(x) = -x^2 + 7x + 30$.

(a)

$$y = -(x^2 - 9x) + 10 = -(x - \frac{9}{2})^2 + \frac{81}{4} + 10 == -(x - \frac{9}{2})^2 + 30.25$$

or

$$y = -(x^2 - 7x) + 30 = -(x - \frac{7}{2})^2 + \frac{49}{4} + 30 == -(x - \frac{7}{2})^2 + 42.25$$

(b) From (a), we directly know by the completing square that the maximum height is 30.25 ft **or** 42.25 ft.

(c) We just need to find out the height of the stone at $x = 9$ and compare with the height 11 feet of the tree. When $x = 9$, by $y = f(x) = -x^2+9x+10$, we know that the stone is at the height of $-9^2+9\cdot 9+10 = 10$ feet which is below the tree, so it hits the tree. **Or**, by $y = f(x) = -x^2 + 7x + 30$, we know that the stone is at the height of $-9^2 + 7 \cdot 9 + 30 = 12$ feet which is above the tree, so it wouldn't hit the tree.