Math 120A - Spring 2004
Mid-Term Exam Number Two
May 20, 2004
Solutions

1. Suppose Tina is considering training to be a 10 kilometer(km) runner. If she trains 0 hours (i.e., if she doesn't train), she can do a 10 km run in 70 minutes. If she trains for 100 hours, she will be able to run 10 km in 60 minutes. If she trains for 300 hours, she will be able to run 10 km in 52 minutes. Suppose the time it takes her to run 10 km is a linear-to-linear rational function of the number of hours that she trains. With an unlimited amount of training, how fast could she possibly run 10 km?

   Solution:

   Let $T(x)$ be the time it takes Tina to run 10km if she has trained x hours. Then

   $$T(x) = \frac{ax+b}{x+c}$$

   for some constants a, b and c. Since $T(0) = 70$, $T(100) = 60$ and $T(300) = 52$, we have the equations

   $$T(0) = 70 = \frac{b}{c} \tag{1}$$

   $$T(100) = 60 = \frac{100a+b}{100+c} \text{ or } 6000 + 60c = 100a + b \tag{2}$$

   $$T(300) = 52 = \frac{300a+b}{300+c} \text{ or } 15600 + 52c = 300a + b \tag{3}$$

   From equation (1), we have $b = 70c$, so we can rewrite equations (2) and (3):

   $$6000 = 100a + 10c \tag{4}$$

   $$15600 = 300a + 18c \tag{5}$$

   By subtracting equation (5) from three times equation (4) we have

   $$2400 = 12c$$

   so that $c = 200$. From this we have $b = 14000$ and $a = 40$. So

   $$T(x) = \frac{40x + 14000}{x + 200}.$$

   $T(x)$ has $y = 40$ as its horizontal asymptote: with an unlimited amount of training, Tina could run the 10km in 40 minutes.

2. A spacecraft landed on another planet. The atmosphere outside the spacecraft was hot, and the temperature increased until it reached a maximum of 180° C four hours after the landing. It then started to decrease, reaching a minimum of -10° C forty hours after the landing.

   Assume that the temperature is a sinusoidal function of the time since the landing.

   The astronauts can go outside of the spacecraft when the temperature is below 50° C. For how many hours during each period of the function is the temperature below 50° C?

   Solution:

   We want to find sinusoidal function $h(t)$ that gives the temperature t hours after landing. We can assume

$$h(t) = A\sin(\frac{2\pi}{B}(t - C)) + D$$

   for constants A, B, C and D. Then:

$$A = \frac{180 - (-10)}{2} = 95$$

$$D = \frac{180 + (-10)}{2} = 85$$

$$B = 2(40 - 4) = 2(36) = 72$$

$$C = 4 - \frac{B}{4} = 4 - \frac{72}{4} = -14$$

   so

$$h(t) = 95\sin(\frac{2\pi}{72}(t + 14)) + 85$$

   Setting this equal to 50 and solving for t, we find

$$t = -18.32365448443162973$$

   The next time the temperature was 50 degrees is a symmetry solution of this solution, so it is

$$4 + (4 - (-18.32365448443162973)) = 26.32365448443162973 2683119$$

   Between these two solutions, the temperature is ≥ 50 degrees. The length of time then per period during which the temperature is at least 50 degrees is

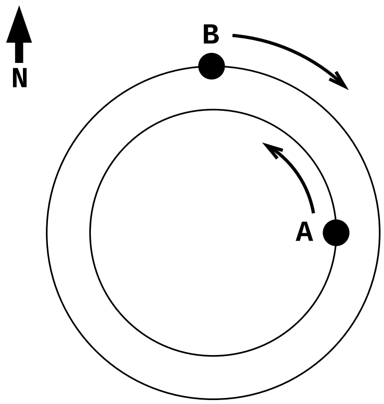
$$26.323654484431629732683119 - (-18.32365448443162973) = 44.6473089688 \text{ hours}$$

   so that

$$72 - 44.6473089688 = 27.3526910311367 \text{ hours}$$

   of each period is below 50◦ C.

3. Agnes and Boris are running around in circles with the same center. They start at the same time from locations as illustrated in the figure: Boris at the northernmost point of his circular path, Agnes at the easternmost point of her path. Agnes runs counter-clockwise at 10 feet per second, and her path has a radius of 200 feet. Boris runs clockwise at 9 feet per second, and his path has a radius of 240 feet.



How far apart are Agnes and Boris after they have been running for 5 minutes?

Solution:

Using the relationship

$$v = r\omega$$

we find that Agnes' angular velocity is 0.05 rad/sec and Boris' angular velocity is 0.0375 rad/sec. In 5 minutes, which is 300 seconds, then Agnes moves through

$$300(0.05) = 15 \text{ radians}$$

while Boris moves through

$$300(0.0375) = 11.25 \text{ radians.}$$

Using the standard xy-coordinate axes, with the origin at the center of the circles, Agnes has moved counterclockwise from an angle of 0 through an angle of 15 radians, so her position is given by

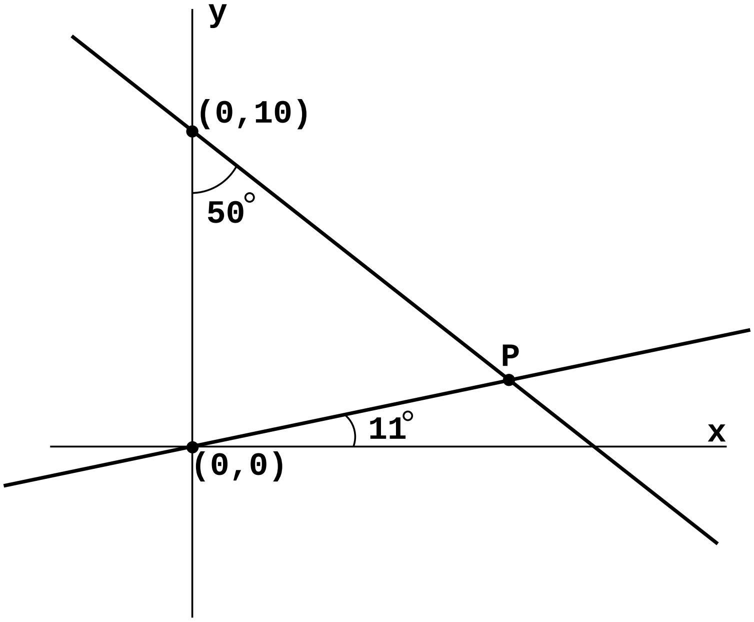
$$(200\cos 15, 200\sin 15) = (-151.9375, 130.05756)$$

Boris has moved clockwise 11.25 radians from an angle of $\pi/2$ to end in the location

$$(240\cos(\frac{\pi}{2} - 11.25), 240\sin(\frac{\pi}{2} - 11.25)) = (-232.273919, 60.405516)$$

The distance formula finds the distance between these two points to be 106.3265506 feet.

4. Find the coordinates of point P in the figure below.



Solution:

Let $P = (x, y)$. Then

$$\tan 11^\circ = \frac{y}{x}$$

and

$$\tan 50^\circ = \frac{x}{10 - y}$$

Solving for y, we find

$$y = \frac{10 \tan 50^\circ \tan 11^\circ}{1 + \tan 50^\circ \tan 11^\circ} = 1.8808329175549602$$

and $x = 9.676046539376526159$.