

Math 120A - Spring 2004
Mid-Term Exam Number One
April 22, 2004
Solutions

1. A train runs along a straight track from the city of Springfield to the city of Alberta. Alberta is located 40 miles south and 35 miles east of Springfield.

(a) The tiny nation of Circlvania is an 18 mile radius circle centered 20 miles south of Springfield. For what distance of the train's trip will the train be in Circlvania?

Solution: We first want to impose a coordinate system. There are several reasonable choices. One is to put the center of the circle (i.e., the center of Circlvania) at the origin $(0, 0)$. Then Springfield is at the point $(0, 20)$, and Alberta is at $(35, -20)$.

The train's track is the line passing through $(0, 20)$ and $(35, -20)$. This line has slope

$$m = \frac{20 - (-20)}{0 - 35} = \frac{40}{-35} = -\frac{8}{7}$$

and so it has equation

$$y = -\frac{8}{7}x + 20.$$

The circle that is the border of Circlvania has equation

$$x^2 + y^2 = 18^2 = 324$$

We are interested in the intersection of the line with the circle, so we solve the equation

$$x^2 + \left(-\frac{8}{7}x + 20\right)^2 = 324$$

for x . Expanding and simplifying, we have

$$\frac{113}{49}x^2 - \frac{320}{7}x + 76 = 0$$

so, using the quadratic formula, we have

$$x = \frac{\frac{320}{7} \pm \sqrt{\left(\frac{320}{7}\right)^2 - 4\left(\frac{113}{49}\right)(76)}}{2\frac{113}{49}}$$

so we can conclude that our two solutions are

$$x = 1.831766325680 \text{ and } x = 17.99124252387699.$$

These are the x -coordinates of the points where the train enters and exits (respectively) Circlvania. The y -values of these points are

$$y = 17.906552770650 \text{ and } y = -0.5614200272879$$

and the distance between the two points (i.e., the distance that the train travels in Circlvania) is

$$\begin{aligned} & \sqrt{(1.8317663256 - 17.991242523876)^2 + (17.906552770 - (-0.56142002728))^2} \\ & = 24.5396554634 \text{ miles.} \end{aligned}$$

(b) How close does the train get to the center of Circlvania?

Solution: Using the same coordinate system as in part (a), we construct a line perpendicular to the train's track through the origin. Since the track is represented by the line

$$y = -\frac{8}{7}x + 20$$

the perpendicular line that passes through the origin is

$$y = \frac{7}{8}x.$$

The intersection of these lines is the nearest point on the track to the origin:

$$\begin{aligned} -\frac{8}{7} + 20 &= \frac{7}{8}x \\ 20 &= \left(\frac{7}{8} + \frac{8}{7}\right)x = \frac{113}{56}x \\ x &= \frac{1120}{113}. \end{aligned}$$

The y -coordinate of this point is

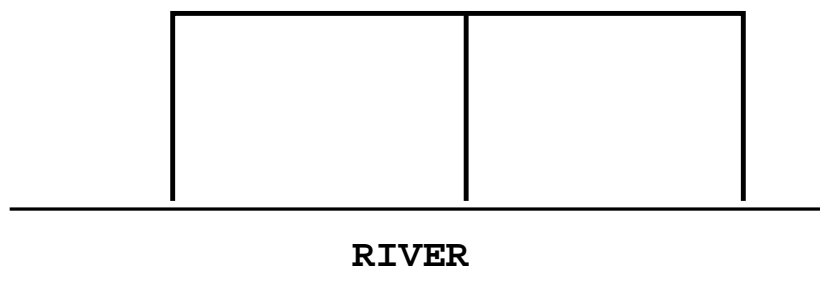
$$y = \frac{7}{8} \frac{1120}{113} = \frac{980}{113}$$

and the distance from this point to the center of Circlvania (i.e., the origin) is

$$\sqrt{\left(\frac{1120}{113}\right)^2 + \left(\frac{980}{113}\right)^2} = 13.17009215 \text{ miles.}$$

2. A farmer wants to make a rectangular enclosure with 220 meters of fencing. One side of the rectangle is bounded by a river, so no fencing is needed on that side of the rectangle. Also, she wants to use some fencing to split the enclosure into two compartments as shown in the figure.

What dimensions should she use for the enclosure to get the maximum possible area?



Solution: Let x be the length of a vertical side of the enclosure and y be the length of a horizontal side. Then the quantity we want to make as large as possible is the area

$$A = xy$$

Now, the amount of fencing needed to make the enclosure is

$$3x + y$$

so

$$220 = 3x + y$$

Solving for y , we have

$$y = 220 - 3x$$

so we can rewrite

$$A = xy = x(220 - 3x) = -3x^2 + 220x$$

Note that this is a quadratic function with a negative leading coefficient, so its maximum value will occur at its vertex. To find the vertex, we can put it into vertex form by completing the square:

$$A = -3x^2 + 220x = -3 \left(x^2 - \frac{220}{3}x \right) = -3 \left(\left(x - \frac{110}{3} \right)^2 - \left(\frac{110}{3} \right)^2 \right) = -3 \left(\left(x - \frac{110}{3} \right)^2 \right) + 3 \left(\frac{110}{3} \right)^2$$

From this, we can see that the largest area the enclosure can have is $3 \left(\frac{110}{3} \right)^2 = \frac{12100}{3} = 4033.333\dots$ square meters, and this will happen with dimensions

$$x = \frac{110}{3} = 36.6666\dots \text{ meters and } y = 110 \text{ meters.}$$

3. Let $f(x) = 5x^2 - 1$.

(a) Find the expression for

$$\frac{f(x+h) - f(x)}{h}$$

and simplify as much as possible.

Solution:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{5(x+h)^2 - 1 - (5x^2 - 1)}{h} = \frac{5(x^2 + 2xh + h^2) - 1 - 5x^2 + 1}{h} = \\ &= \frac{5x^2 + 10xh + 5h^2 - 1 - 5x^2 + 1}{h} = \frac{10xh + 5h^2}{h} = 10x + 5h. \end{aligned}$$

(b) Let $g(x) = x + 2$. Find all solutions to the equation $g(f(x)) = f(g(x))$.

Solution:

$$g(f(x)) = f(x) + 2 = 5x^2 - 1 + 2 = 5x^2 + 1$$

and

$$f(g(x)) = f(x+2) = 5(x+2)^2 - 1 = 5(x^2 + 4x + 4) - 1 = 5x^2 + 20x + 20 - 1 = 5x^2 + 20x + 19$$

so if

$$f(g(x)) = g(f(x))$$

then

$$5x^2 + 1 = 5x^2 + 20x + 19$$

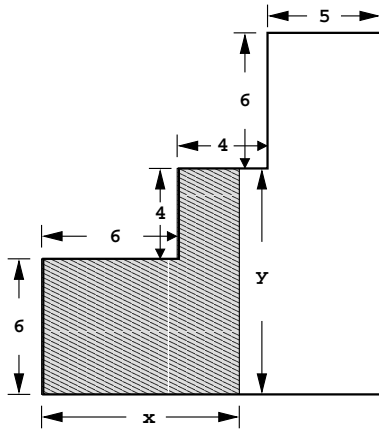
which simplifies to

$$-18 = 20x$$

or

$$x = \frac{-18}{20} = -\frac{9}{10}.$$

4. Pizzeria Stairsteppo makes a pizza shaped as shown below. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Let x be the bottom length of Alice's portion and y be the length of the cut as shown in the figure.



- (a) Find a formula for y as a multipart function of x , for $0 \leq x \leq 15$.

Solution:

$$y = \begin{cases} 6 & \text{if } 0 \leq x \leq 6, \\ 10 & \text{if } 6 < x \leq 10, \\ 16 & \text{if } 10 < x \leq 15 \end{cases}$$

The value of y at $x = 6$ could instead be 10, and the value of y at $x = 10$ could instead be 16, so there are several acceptable answers.

- (b) Find a formula for the area of Alice's portion as a multipart function of x , for $0 \leq x \leq 15$.

Solution:

$$\begin{aligned} \text{area} &= \begin{cases} 6x & \text{if } 0 \leq x \leq 6, \\ 36 + 10(x - 6) & \text{if } 6 < x \leq 10, \\ 76 + 16(x - 10) & \text{if } 10 < x \leq 15 \end{cases} \\ &= \begin{cases} 6x & \text{if } 0 \leq x \leq 6, \\ 10x - 24 & \text{if } 6 < x \leq 10, \\ 16x - 84 & \text{if } 10 < x \leq 15 \end{cases} \end{aligned}$$